

# ON MULTIPLICATIVE FUNCTIONS WHICH ARE SMALL ON AVERAGE

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**ABSTRACT.** Let  $f$  be a completely multiplicative function that assumes values inside the unit disc. Halász showed that unless  $f$  *pretends to be*  $n^{it}$  for some fixed  $t$ , then  $f$  has mean value 0, and gave a quantitative estimate for the rate of decay of the partial sums of  $f$ . Due to their generality, these estimates are weak quantitatively. The goal of this paper is to study for which functions  $f$  it is possible to improve upon the bound supplied by Halász's theorem. As a starting point, we observe that if the values of  $f$  at primes have some regularity, then  $f$  can exhibit a lot of cancelation on average only if either  $f(p)$  is small on average or  $f$  pretends to be  $\mu(n)n^{it}$  for some  $t$ . Inverting this observation, we show that it is possible to improve Halász's theorem exponentially once we restrict ourselves to a suitable class of completely multiplicative functions.

## CONTENTS

1. Introduction	1
1.1. Outline of the paper	6
1.2. Notation	6
2. Main results	6
3. Preliminaries	9
4. Bounds for $L(s, f)$	12
5. Bounds for $\frac{1}{L}(s, f)$ and $\frac{L'}{L}(s, f)$	16
6. Distances of multiplicative functions	18
7. Real zeroes and the size of $L(1, f)$	24
8. Proof of Theorems 1.2 and 2.1	26
8.1. Technical preparation	26
8.2. Two intermediate results	33
8.3. Completion of the proofs	38
Acknowledgements	41
References	42

## 1. INTRODUCTION

A multiplicative function is an arithmetic function  $f : \mathbb{N} \rightarrow \mathbb{C}$  which satisfies the functional equation  $f(mn) = f(m)f(n)$  whenever  $(m, n) = 1$ . Many problems in number theory can be phrased in terms of the average behavior of multiplicative functions. A question of particular importance is when a given a multiplicative function  $f$  has mean value 0. This

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problem was solved by Halász [Ha71, Ha75] when  $f$  assumes values inside the unit circle  $\mathbb{U} = \{z \in \mathbb{C} : |z| \leq 1\}$ . His result states that unless  $f$  *pretends to be*  $n^{it}$  for some  $t \in \mathbb{R}$ , in the sense that

$$\sum_p \frac{1 - \Re(f(p)p^{-it})}{p} < \infty,$$

then  $f$  is 0 on average; the converse is also true. Halász also gave a quantitative version of his result and various authors ([M78], [GS03], [T]) improved on it. The state of the art on this problem is Theorem 1.1 below. Here and for the rest of this paper, given two multiplicative functions  $f, g : \mathbb{N} \rightarrow \mathbb{U}$  and  $x \geq y \geq 1$ , we set

$$\mathbb{D}(f, g; y, x) = \left( \sum_{y < p \leq x} \frac{1 - \Re(f(p)\overline{g(p)})}{p} \right)^{1/2}.$$

This quantity measures a certain “distance” between  $f$  and  $g$ ; as a matter of fact, it satisfies the triangle inequality (see Lemma 6.3).

**Theorem 1.1.** *Let  $f : \mathbb{N} \rightarrow \mathbb{U}$  be a multiplicative function and consider  $x \geq 2$  and  $T \geq 1$ . Then we have that*

$$\frac{1}{x} \sum_{n \leq x} f(n) \ll \frac{M_f(x; T) + 1}{e^{M_f(x; T)}} + \frac{1}{T}, \quad \text{where} \quad M_f(x; T) = \min_{|t| \leq T} \mathbb{D}^2(f(n), n^{it}; 1, x).$$

The generality of the above theorem is quite striking as it makes no assumptions for  $f$  other than that its range of values is  $\mathbb{U}$ . Nevertheless, the breadth of applicability of Theorem 1.1 comes at a price: it can be shown that  $M_f(x, T) \leq \log \log x + O(1)$  ([GS]), so the best bound on  $\sum_{n \leq x} f(n)$  that Theorem 1.1 can yield is  $cx \log \log x / \log x$ , where  $c$  is some absolute constant. In the converse direction, Montgomery and Vaughan [MV01] constructed for every  $x \geq 2$  a multiplicative function whose partial sum up to  $x$  is of size  $x \log \log x / \log x$ , thus showing that Theorem 1.1 is best possible. More recently, Granville and Soundararajan [GS03] showed an explicit version of Theorem 1.1 and constructed multiplicative functions whose summatory function achieves the bound in [GS03] within a factor of 10. It is not very hard to construct slightly weaker but still almost extremal examples. Indeed, the completely multiplicative function  $f$  defined by

$$(1.1) \quad f(p) = \begin{cases} 1 & \text{if } x/2 < p \leq x, \\ 0 & \text{otherwise,} \end{cases}$$

satisfies the estimates

$$\sum_{n \leq x} f(n) = \sum_{x/2 < p \leq x} 1 \sim \frac{x}{2 \log x} \quad \text{and} \quad M_f(x, T) = \log \log x + O(1).$$

Even though Theorem 1.1 is optimal in this general setting, there are specific multiplicative functions for whose partial sums we know or conjecture much sharper estimates than  $\ll x \log \log x / \log x$ . An important example is the Möbius function, since controlling the size of the partial sums of the Möbius function corresponds to estimating the error term in the prime number theorem.

In order to understand the limitations of Halász's theorem better we study the following question: which multiplicative functions  $f$  satisfy the relation

$$(1.2) \quad \sum_{n \leq x} f(n) \ll_A \frac{x}{(\log x)^A} \quad \text{for all } x \geq 2,$$

for some constant  $A > 2$ ? For simplicity and in order to avoid technical issues at the prime 2, we assume further that  $f$  is completely multiplicative (see Remark 1.1 for further discussion about this). The key observation towards understanding this problem is that if  $f(p)$  is equal to  $v \in \mathbb{U}$  on average, then by the Selberg-Delange method [T, Chap. II.6] we expect that

$$(1.3) \quad \sum_{n \leq x} f(n) = \left( \frac{c_{f,v}}{\Gamma(v)} + o(1) \right) x (\log x)^{v-1} \quad (x \rightarrow \infty),$$

where  $c_{f,v}$  is some non-zero constant and  $\Gamma$  denotes Euler's Gamma function. Therefore, unless  $v$  is a pole of  $\Gamma$ , relation (1.2) cannot hold for any  $A > 2 \geq 1 - \Re(v)$ . The only poles of  $\Gamma$  in the unit circle are located at  $-1$  and at  $0$ . If now  $v = -1$ , then  $f$  looks like the Möbius function  $\mu$  which satisfies (1.2) by a quantitative form of the prime number theorem. Lastly, for the case  $v = 0$  Granville [GS] showed that

$$(1.4) \quad \sum_{p \leq x} f(p) \log p \ll \frac{x}{(\log x)^B} \quad (x \geq 2) \quad \implies \quad \sum_{n \leq x} f(n) \ll \frac{x}{(\log x)^B} \quad (x \geq 2).$$

The above remarks seem to suggest that if (1.2) holds, then the mean value of  $f(p)$  has to be  $-1$  or  $0$ . However, this is obviously false, as the completely multiplicative function  $(-1)^{\Omega(n)} n^{it}$  also satisfies (1.2) by the prime number theorem<sup>1</sup>. We make the refined guess that if (1.2) holds, then either  $f$  pretends to be  $\mu(n) n^{it}$  for some  $t$  or  $f(p)$  is  $0$  on average. The following theorem confirms partially our guess.

**Theorem 1.2.** *Let  $f : \mathbb{N} \rightarrow \mathbb{U}$  be a completely multiplicative function that satisfies (1.2) for some  $A \geq 3$ .*

(a) *We have that*

$$\left| \frac{1}{x} \sum_{n \leq x} f(n) \mu(n) \right| + \left| \frac{1}{x} \sum_{p \leq x} f(p) \log p \right| \ll_A \frac{(M+1)^{1/2}}{e^{(A-3)M/2}},$$

where

$$\begin{aligned} M &= \min_{|t| \leq (\log x)^{A-2}} \left\{ \frac{\log(1+|t|)}{A-2} + \mathbb{D}^2 \left( f(n), \mu(n) n^{it}; \exp \left\{ (1+|t|)^{\frac{1}{A-2}} \right\}, x \right) \right\} \\ &\geq \frac{M_{\mu f}(x; (\log x)^{A-2})}{2} + O(1). \end{aligned}$$

(b) *If  $\liminf_{x \rightarrow \infty} \sum_{p \leq x} \Re(f(p) p^{-it_0}) / p = -\infty$  for some  $t_0 \in \mathbb{R}$  and  $x \geq \exp\{2(1+|t_0|)^{\frac{1}{A-2}}\}$ , then*

$$\frac{1}{x} \sum_{p \leq x} (f(p) p^{-it_0} + 1) \log p \ll_A \frac{(1+|t_0|)^{\frac{A-3}{2(A-2)}}}{(\log x)^{\frac{A-4}{2}}} \log \left( \frac{\log x}{(1+|t_0|)^{\frac{1}{A-2}}} \right).$$

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<sup>1</sup> $\Omega(n)$  denotes the number of prime divisors of  $n$ , counted with multiplicity.

*Remark 1.1.* It is possible to extend Theorem 1.2 to the class of multiplicative functions  $f : \mathbb{N} \rightarrow \mathbb{U}$ , but we need a stronger assumption on  $f$  than (1.2) that excludes a certain type of behavior of  $f$  on powers of 2. To see that this is necessary, set  $f(n) = 1$  when  $n$  is odd and  $f(n) = -1$  when  $n$  is even. Then  $f$  is multiplicative and  $\sum_{n \leq x} f(n) = O(1)$ . However, the conclusion of Theorem 1.2(a) is clearly false.

In order to avoid the above example, we impose the stronger condition that

$$(1.5) \quad \sum_{\substack{n \leq x \\ (n, 2) = 1}} f(n) \ll \frac{x}{(\log x)^A} \quad (x \geq 2),$$

which is clearly satisfied if  $f$  is completely multiplicative and (1.2) holds. Under condition (1.5), Theorem 1.2 remains true. Indeed, set  $\tilde{f}(n) = f(n)$  if  $(n, 2) = 1$  and  $\tilde{f}(n) = 0$  otherwise. Also, let  $g(n) = \prod_{p^a \parallel n} \tilde{f}(p)^a$  and write  $g = \tilde{f} * h$ , so that  $h$  is supported on square-full integers and it satisfies the bound  $|h(n)| \leq \tau_3(n)$  for all  $n$ . So  $\sum_{n \leq x} |h(n)| \leq \sum_{a^2 b^3 \leq x} \tau_3(a^2 b^3) \ll \sqrt{x} \log^2 x$ . Consequently,  $g$  satisfies (1.2), which allows us to apply Theorem 1.2 to  $g$ . In order to switch back to  $f$ , note that if  $\mu f = \mu g * h'$ , then  $h'$  is supported on  $\{2^r \cdot m : r \geq 0, (m, 2) = 1, m \text{ square-full}\}$  and it satisfies the bound  $|h(n)| \leq \tau_2(n)$  for all  $n$ .

Similar extensions can be made to all subsequent results.

We shall show Theorem 1.2 in Section 8. The starting point of our argument is an idea used by Iwaniec and Kowalski [IK, p. 40-42] to give a new proof of the prime number theorem, which we improve by combining it with some ideas from sieve methods. Note that part (a) of Theorem 1.2 constitutes an improvement over Halász's theorem for a certain class of multiplicative functions. More precisely, if  $f$  satisfies (1.2) for some  $A > 7$ , then Theorem 1.2 goes beyond what Theorem 1.1 can give for the summatory function of  $\mu \cdot f$ .

When  $f$  is real valued, it is possible to exclude the possibility that  $f$  looks like  $\mu(n)n^{it}$  for some  $t \neq 0$  (see Theorem 2.4(b) below) and simplify the statement of Theorem 1.2:

**Corollary 1.3.** *Let  $f : \mathbb{N} \rightarrow [-1, 1]$  be a completely multiplicative function that satisfies (1.2) for some  $A \geq 3$ . Then for  $x \geq 2$  we have that*

$$\begin{cases} \sum_{p \leq x} f(p) \log p \ll_{A,f} \frac{x(\log \log x)^{1/2}}{(\log x)^{\frac{A-3}{2}}} & \text{if } \liminf_{y \rightarrow \infty} \sum_{p \leq y} \frac{f(p)}{p} > -\infty, \\ \sum_{p \leq x} (1 + f(p)) \log p \ll_A \frac{x \log \log x}{(\log x)^{\frac{A-4}{2}}} & \text{if } \liminf_{y \rightarrow \infty} \sum_{p \leq y} \frac{f(p)}{p} = -\infty; \end{cases}$$

in the first case, the implied constant depends on  $f$  via the size of  $\sum_{n \geq 1} f(n)/n$ .

Even though Theorem 1.2 provides a partial answer to our initial question about when (1.2) holds, in practice it is not as useful in proving prime number theorems as one would hope for. The reason is that the partial sums of many interesting multiplicative functions, such as Dirichlet characters or normalized Fourier coefficients of Hecke eigencuspforms, exhibit cancelation only past a certain point, which is related to what is called the analytic conductor of the associated  $L$ -function (see [IK, Chapter 5] for more about this). In the next section we combine the method leading to Theorem 1.2 with some additional ideas, some of which go back to Halász and some of which are novel, to show an explicit version of Theorem 1.2 which takes into account the possible presence of a conductor. Our main result, Theorem 2.1 below,

shows that if, for any  $t \in \mathbb{R}$ , the distance of  $f(n)$  from  $\mu(n)n^{it}$  is big, then  $\sum_{p \leq x} f(p) \log p$  is very small for  $x$  as small as a power of the conductor, in accordance with the classical results from the theory of  $L$ -functions. For now, we state our result in a special case, which is of particular interest; it improves Halász's theorem exponentially for certain multiplicative functions.

**Theorem 1.4.** *Let  $f : \mathbb{N} \rightarrow \mathbb{U}$  be a completely multiplicative function,  $Q \geq 3$  and  $\delta > 0$ . If*

$$\left| \sum_{n \leq x} f(n) \right| \leq x^{1-\delta} \quad (x \geq Q),$$

*then for  $x \geq Q$  we have that*

$$\left| \frac{1}{x} \sum_{n \leq x} f(n) \mu(n) \right| + \left| \frac{1}{x} \sum_{p \leq x} f(p) \log p \right| \ll e^{-c\sqrt{\eta \log x}} + x^{-c\eta/\log Q}$$

*for some  $c = c(\delta)$ , where*

$$\begin{aligned} \eta &= \min_{|t| \leq \sqrt{x}} \prod_{Q+|t| < p \leq x} \left| 1 + \frac{f(p)}{p^{1+it}} \right| \\ &\asymp \frac{\exp \left\{ \min_{|t| \leq \sqrt{x}} \left\{ \log \log(Q + |t|) + \mathbb{D}^2(f(n), \mu(n)n^{it}; Q + |t|, x) \right\} \right\}}{\log x} \ll_{\delta} 1. \end{aligned}$$

Under certain assumptions we can relate the size of the parameter  $\eta$  to the location of zeroes of the associated  $L$ -function (see Theorem 2.5). In particular, if we combine Theorem 1.4 with Theorems 2.4 and 2.5 when  $f$  is a Dirichlet character, then we obtain results as strong as the classical estimates of de la Vallée Poussin [D]. In [K] we showed how to also insert estimates for exponential sums due to Korobov and Vinogradov to our arguments to deduce the best result that is known about the counting function of prime numbers in arithmetic progressions.

We conclude this section with three open problems:

- Together with relation (1.4), Theorem 1.2 allows us to go back and forth between estimates for  $\sum_{n \leq x} f(n)$  and for  $\sum_{p \leq x} f(p) \log p$ . It would be interesting to examine the precise quantitative relation between these two sums.
- It would be desirable to extend the results of this paper to multiplicative functions that assume values outside the unit circle too. A large portion of the paper can be generalized to multiplicative functions whose values at primes are uniformly bounded. However, the results of Section 6 cannot be transferred immediately.
- It is natural to ask what happens if (1.2) holds for some  $A \leq 3$ , since Theorem 1.2 does not cover this range. It is worth noticing here that if  $1 < A < 2$ , then the completely multiplicative function  $f(n) = (1 - A)^{\Omega(n)}$  satisfies (1.2), by (1.3), but violates our guess: neither is  $f(p)$  0 on average nor does  $f$  pretend to be  $\mu(n)n^{it}$  for some  $t \in \mathbb{R}$ . Similarly, when  $A \leq 1$ , the function given by (1.1) provides a counterexample to our guess. So the case that remains open is when  $A \in [2, 3]$ .

**1.1. Outline of the paper.** In Section 2 we state our main technical results, Theorems 2.1, 2.2, 2.3, 2.4 and 2.5. Section 3 contains a series of auxiliary estimates we will be needing throughout the paper. Subsequently, in Sections 4 and 5, we derive bounds for high derivatives of  $L(s, f)$ ,  $\frac{1}{L}(s, f)$  and  $\frac{L'}{L}(s, f)$  close to the line  $\Re(s) = 1$ , which will be crucial in our arguments. In Section 6 we state and prove several results related to distance of a multiplicative function from the Möbius function and apply them to control the size of  $L(s, f)$  close to the line  $\Re(s) = 1$ . In particular, we prove Theorems 2.2, 2.3 and 2.4. Furthermore, in Section 7 we see how to control the size of  $L(1, f)$  in terms of a potential *Siegel zero* and demonstrate Theorem 2.3. Finally, the proof of Theorem 1.2 and Theorem 2.1 is given in Section 8 and it is divided in three parts: In Subsection 8.1 we show some auxiliary results, which are then used in Subsection 8.2 to establish two important intermediate results, Theorems 8.5 and 8.6. Finally, the proof of Theorems 1.2 and 2.1 is completed in Subsection 8.3.

**1.2. Notation.** For an integer  $n$  we denote with  $P^+(n)$  and  $P^-(n)$  the greatest and smallest prime divisors of  $n$ , respectively, with the notational convention that  $P^+(1) = 1$  and  $P^-(1) = \infty$ . For two arithmetic functions  $f, g : \mathbb{N} \rightarrow \mathbb{C}$  we write  $f * g$  for their Dirichlet convolution, defined by  $f * g(n) = \sum_{ab=n} f(a)g(b)$ . Also, for  $y \geq 1$  and  $s \in \mathbb{C}$  we set

$$L(s, f) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \quad \text{and} \quad L_y(s, f) = \sum_{P^-(n) > y} \frac{f(n)}{n^s},$$

provided that the series converge. In the special case that  $f(n) = 1$  for all  $n$ , we use the notation

$$\zeta_y(s) = L_y(s, 1).$$

We let  $\tau_k(n) = \sum_{d_1 \dots d_k = n} 1$  and we denote with  $\mu(n)$  the Möbius function, defined to be  $(-1)^{\#\{p|n\}}$  if  $n$  is squarefree and 0 otherwise. Moreover, we recall the definition of the generalized von Mangoldt functions  $\Lambda_k = \mu * \log^k$ . The case  $k = 1$  corresponds to the regular von Mangoldt function, which we denote simply by  $\Lambda$ ; its value at an integer  $n$  is  $\log p$  if  $n$  is a prime power  $p^a$  and 0 otherwise. Finally, the notation  $F \ll_{a,b,\dots} G$  means that  $|F| \leq CG$ , where  $C$  is a constant that depends at most on the subscripts  $a, b, \dots$ , and  $F \asymp_{a,b,\dots} G$  means that  $F \ll_{a,b,\dots} G$  and  $G \ll_{a,b,\dots} F$ . In general, we reserve the letters  $c$  and  $C$  in order to denote constants, not necessarily the same ones in every place, and possibly depending on certain parameters that will be specified using subscripts and other means.

## 2. MAIN RESULTS

In this section we state the main results of the paper. First, we have the following theorem which is an explicit version of Theorem 1.2.

**Theorem 2.1.** *Let  $f : \mathbb{N} \rightarrow \mathbb{U}$  be a completely multiplicative function,  $Q \geq 3$ ,  $A \geq 3$  and  $\delta > 0$  such that*

$$(2.1) \quad \left| \sum_{n \leq x} f(n) \right| \leq \frac{(\log Q)^{A-2}}{(\log x)^A} + x^{1-\delta} \quad (x \geq Q).$$

For  $t \in \mathbb{R}$  define  $Q_t = Q_t(A, \delta, Q)$  via

$$(2.2) \quad \log Q_t = \max \left\{ 2(1 + |t|)^{\frac{1}{A-2}} \log Q, \frac{4 \log(2 + |t|)}{\delta} \right\}.$$

Consider  $x \geq 2$ ,  $T \geq 1$  and  $k \in \mathbb{Z} \cap [0, (A-3)/2]$ .

(a) We have that

$$(2.3) \quad \frac{1}{x} \sum_{n \leq x} f(n) \mu(n) \ll c^k k! \left\{ \left( \frac{k + \log Q}{e^{N(x;T)}} \right)^{k + \frac{1}{2} - \frac{(k+1)(k+2)}{4(A-1)}} + \frac{1}{\sqrt{T}} \right\},$$

where  $c = c(\delta)$  is some constant and  $N_f(x; T) = N_f(x; T; A, \delta, Q)$  is given by

$$(2.4) \quad N_f(x; T) = \min_{\substack{|t| \leq T \\ Q_t \leq x}} \{ \log \log Q_t + \mathbb{D}^2(f(n), \mu(n)n^{it}; Q_t, x) \} \geq \frac{M_{\mu f}(x; T)}{2} + O(1).$$

If, in addition,  $1 \leq k \leq (A-3)/2$ , then

$$(2.5) \quad \frac{1}{x} \sum_{p \leq x} f(p) \log p \ll c^k k! \left\{ \left( \frac{k + \log Q}{e^{N(x;T)}} \right)^{k - \frac{1}{2} - \frac{(k+1)(k+2)}{4(A-1)}} + \sqrt{\frac{\log x}{T}} \right\},$$

(b) If  $1 \leq k \leq (A-3)/2$  and  $\liminf_{x \rightarrow \infty} \sum_{p \leq x} \Re(f(p))/p = -\infty$ , then we have that

$$\frac{1}{x} \sum_{p \leq x} (f(p) + 1) \log p \ll (ck)^k \left( \frac{\log Q + k}{\log x} \right)^{k - \frac{1}{2} - \frac{(k+1)(k+2)}{4(A-1)}}.$$

Note that Theorem 1.4 follows immediately by applying Theorem 2.1 with  $A = \infty$  and  $T = \infty$  and then choosing  $k \asymp \min\{e^{N_f(x; \infty)/2}, e^{N_f(x; \infty)}/\log Q\}$ , since  $e^{N_f(x; \infty)} \asymp \eta \log x$  by Lemma 6.1 below.

A key role in the proof of Theorems 1.2 and 2.1 is played by the following two theorems, which are of independent interest. Their proof will be given in Section 6.

**Theorem 2.2.** Let  $Q \geq 2$ ,  $\epsilon > 0$  and  $f : \mathbb{N} \rightarrow \mathbb{U}$  a completely multiplicative function such that

$$(2.6) \quad \left| \sum_{n \leq x} f(n) \right| \leq \frac{x(\log Q)^\epsilon}{(\log x)^{2+\epsilon}} \quad (x \geq Q).$$

Then there is some  $Q' \in [Q, +\infty]$  such that

$$\mathbb{D}(f(n), \mu(n); Q, Q') \ll_\epsilon 1 \quad \text{and} \quad \sum_{Q' < p \leq z} \frac{f(p)}{p} \ll_\epsilon 1 \quad (z \geq Q').$$

So for  $y \geq Q$  we have that  $|L_y(1, f)| \asymp (\log y)/\log(yQ')$ . In particular, letting  $y = Q$ , we find that  $\log Q' \asymp (\log Q)/|L_Q(1, f)|$ .

**Theorem 2.3.** Let  $f : \mathbb{N} \rightarrow \mathbb{U}$  be a completely multiplicative function,  $Q \geq 3$ ,  $1 \leq \tau \leq \exp \left\{ \frac{(\log Q)^{3/2}}{2000\sqrt{\log \log Q}} \right\}$  and  $\epsilon > 0$  such that

$$\left| \sum_{n \leq x} f(n) n^{-it} \right| \leq \frac{x}{(\log x)^2} \left( \frac{\log Q}{\log x} \right)^\epsilon \quad (x \geq Q, t \in [-\tau, \tau]).$$

Fix  $\sigma > 1$  and  $J \subset [-\tau, \tau]$  and let  $t_0 \in J$  be such that  $|L_Q(\sigma + it_0, f)| = \min_{t \in J} |L_Q(\sigma + it, f)| =: \eta$ . Then, for any  $t \in J$ , we have that

$$|L_Q(\sigma + it, f)| \asymp_\epsilon \begin{cases} \eta & \text{if } |t - t_0| \log Q \leq \eta, \\ |t - t_0| \log Q & \text{if } \eta \leq |t - t_0| \log Q \leq 1, \\ 1 & \text{if } |t - t_0| \log Q \geq 1. \end{cases}$$

If we have additional information about  $f$ , then it is possible to control the size of  $N_f(x; T)$ . This is the context of the following theorem, which will be proven in Section 6.

**Theorem 2.4.** *Let  $Q \geq 3$ ,  $\epsilon > 0$  and  $f : \mathbb{N} \rightarrow \mathbb{U}$  be a completely multiplicative function satisfying (2.6).*

(a) *If  $f^2$  satisfies (2.6) too, then*

$$\sum_{Q < p \leq x} \frac{f(p)}{p} \ll 1 \quad (x \geq Q).$$

(b) *Assume that  $f(p) \in \mathbb{R}$  for  $p > Q$ . Let  $x \geq Q$  and  $t \in \mathbb{R}$  with  $|t| \leq \exp \left\{ \frac{(\log Q)^{3/2}}{2000\sqrt{\log \log Q}} \right\}$ . If  $|t| \geq 1/\log Q$ , then*

$$\sum_{Q < p \leq x} \frac{f(p)}{p^{1+it}} \ll 1,$$

*and if  $|t| \leq 1/\log Q$ , then*

$$\mathbb{D}^2(f(n), \mu(n)n^{it}; Q, x) \geq \mathbb{D}^2(f(n), \mu(n); Q, x) - O(1) \geq \log \left( \frac{\log x}{\log Q} \right) + \log L_Q(1, f) - O(1).$$

Finally, if we have at our disposal very good estimates on the summatory function of  $f$ , then we can show that  $L(s, f)$  converges to the left of the line  $\Re(s) = 1$ , by partial summation. If, in addition,  $f$  is real valued, then the size of  $L_Q(1, f)$  can be determined using information about the location of the zeroes of  $L(s, f)$  when  $s < 1$ . This is the context of the next theorem. Its proof, which will be given in Section 7, is elementary and it uses some ideas of Pintz [Pi76a, Pi76b, Pi76c], who gave elementary proofs of some related results when  $f$  is a real Dirichlet character.

**Theorem 2.5.** *Let  $Q \geq 3$  and  $f : \mathbb{N} \rightarrow \mathbb{U}$  be a completely multiplicative function such that  $f(p) \in \mathbb{R}$  for  $p > Q$  and*

$$\left| \sum_{n \leq x} f(n) \right| \leq \frac{x^{1-1/\log Q}}{\log^2 x} \quad (x \geq Q).$$

*Then  $L(s, f)$  converges in the half plane  $\Re(s) > 1 - 1/\log Q$  and there is an absolute constant  $c \in (0, 1)$  such that  $L(s, f)$  has at most one zero in  $[1 - 2c/\log Q, 1]$ , say at  $\beta$ . If no such zero exists, we set  $\beta = 1 - 2c/\log Q$ . In any case, there are positive constants  $c_1$  and  $c_2$  such that for all  $\sigma \in [1 - c/\log Q, 1 + c/\log Q]$  we have that*

$$c_1(\sigma - \beta) \log Q \leq L_Q(\sigma, f) \leq c_2(\sigma - \beta) \log Q.$$



## 3. PRELIMINARIES

In this section we present a series of auxiliary results we will be using throughout the paper. We start with the following general lemma, which is based on an idea in [IK, p. 40], also exploited in [K, Lemma 2.1].

**Lemma 3.1.** *Let  $k \in \mathbb{N}$ ,  $D \subset \mathbb{C}$  be an open set,  $s \in D$ , and  $F : D \rightarrow \mathbb{C}$  be a function which is differentiable  $k$  times at  $s$ . Assume that  $F(s) \neq 0$  and set*

$$M = \sup_{1 \leq j \leq k} \left\{ \frac{1}{j!} \left| \frac{F^{(j)}(s)}{F}(s) \right| \right\}^{1/j}, \quad N = \sup_{1 \leq j \leq k} \left\{ \frac{1}{j!} \left| \left( \frac{F'}{F} \right)^{(j-1)}(s) \right| \right\}^{1/j}.$$

Then  $M/2 \leq N \leq 8M$ .

*Proof.* By arguing as in [K, Lemma 2.1], we find that  $N \leq 8M$ .

In order to show that  $M \leq 2N$ , we argue inductively. First, we have that  $|(F'/F)(s)| \leq N$ , by the definition of  $N$ . Next, we assume that  $|F^{(j)}(s)/F(s)| \leq j!(2N)^j$  for all  $j \in \{1, \dots, r\}$ , for some  $r \in \{1, \dots, k-1\}$ . Since

$$F^{(r+1)}(s) = \left( F \cdot \frac{F'}{F} \right)^{(r)}(s) = \sum_{j=0}^r \binom{r}{j} F^{(j)}(s) \left( \frac{F'}{F} \right)^{(r-j)}(s),$$

we find that

$$\left| \frac{F^{(r+1)}}{F}(s) \right| \leq (r+1)! \sum_{j=0}^r (2N)^j N^{r-j+1} < (r+1)!(2N)^{r+1},$$

which completes the inductive step, and the lemma follows.  $\square$

The next lemma is due to Montgomery [M94, Theorem 3, p. 131].

**Lemma 3.2.** *Let  $A(s) = \sum_{n \geq 1} a_n/n^s$  and  $B(s) = \sum_{n \geq 1} b_n/n^s$  be two Dirichlet series which converge for  $\Re(s) > 1$ . If  $|a_n| \leq b_n$  for all  $n \in \mathbb{N}$ , then*

$$\int_{-T}^T |A(\sigma + it)|^2 dt \leq 3 \int_{-T}^T |B(\sigma + it)|^2 dt \quad (\sigma > 1, T \geq 0).$$

Also, we need a result which is known as the *fundamental lemma of sieve methods*. It has appeared in the literature in many different forms (for example, see [HR, Theorem 7.2]). The version we shall use is a direct consequence of Lemma 5 in [FI78].

**Lemma 3.3.** *Let  $y \geq 2$  and  $D = y^u$  with  $u \geq 2$ . There exist two arithmetic functions  $\lambda^\pm : \mathbb{N} \rightarrow [-1, 1]$ , supported in  $\{d \in \mathbb{N} : P^+(d) \leq y, d \leq D\}$ , for which*

$$\begin{cases} (\lambda^- * 1)(n) = (\lambda^+ * 1)(n) = 1 & \text{if } P^-(n) > y, \\ (\lambda^- * 1)(n) \leq 0 \leq (\lambda^+ * 1)(n) & \text{else.} \end{cases}$$

Moreover, if  $g : \mathbb{N} \rightarrow \mathbb{R}$  is a multiplicative function with  $0 \leq g(p) \leq \min\{1, p-1\}$  for all primes  $p \leq y$ , then

$$\sum_d \frac{\lambda^\pm(d)g(d)}{d} = (1 + O(e^{-u})) \prod_{p \leq y} \left( 1 - \frac{g(p)}{p} \right).$$

In addition, we need the following simple sieve estimate.

**Lemma 3.4.** *Let  $3/2 \leq y \leq z$  and  $k \in \mathbb{N} \cup \{0\}$ . Then*

$$\sum_{p|n \Rightarrow y < p \leq z} \frac{\log^k n}{n} \ll \frac{k!(3 \log z)^{k+1}}{\log y}.$$

*Proof.* For every  $m \in \mathbb{N}$  we have that

$$\begin{aligned} \sum_{\substack{z^{m-1}-1 < n \leq z^m-1 \\ p|n \Rightarrow y < p \leq z}} \frac{\log^k n}{n} &\ll \frac{(m \log z)^k}{e^{m/3}} \sum_{p|n \Rightarrow y < p \leq z} \frac{1}{n^{1-1/(3 \log z)}} \\ &\leq \frac{(m \log z)^k}{e^{m/3}} \prod_{y < p \leq z} \left( 1 + \frac{1}{p^{1-1/(3 \log z)}} + \frac{1}{p^{2-2/(3 \log z)}} + \cdots \right) \\ &= \frac{(m \log z)^k}{e^{m/3}} \prod_{y < p \leq z} \left( 1 + \frac{1}{p} + O\left( \frac{\log p}{p \log z} + \frac{1}{p^{2-2/(3 \log z)}} \right) \right) \\ &\ll \frac{m^k (\log z)^{k+1}}{e^{m/3} \log y}. \end{aligned}$$

Summing the above inequality over all  $m \in \mathbb{N}$  completes the proof of the lemma.  $\square$

We state below a simple corollary of Lemma 2.4 in [K]. Here and for the rest of the paper we define

$$(3.1) \quad V_t = \exp\{(\log(3 + |t|))^{2/3}(\log \log(3 + |t|))^{1/3}\} \quad (t \in \mathbb{R}).$$

**Lemma 3.5.** *Let  $y \geq 2$  and  $s = \sigma + it$  with  $t \in \mathbb{R}$ ,  $y \geq V_t^{100}$  and  $\sigma \geq 1 - 1/(60 \log y)$ . For  $x \geq y$  we have that*

$$\sum_{\substack{n \leq x \\ P^-(n) > y}} n^{it} = \frac{x^{1+it}}{1+it} \prod_{p \leq y} \left( 1 - \frac{1}{p} \right) + O\left( \frac{x^{1-\frac{1}{30 \log y}}}{\log y} \right)$$

and, consequently,

$$\sum_{\substack{n \leq x \\ P^-(n) > y}} \frac{1}{n^s} = \left( \frac{1 - x^{-s+1}}{s-1} + \gamma_{s,y} \right) \prod_{p \leq y} \left( 1 - \frac{1}{p} \right) + O\left( \sigma x^{1-\sigma-\frac{1}{30 \log y}} \right),$$

where  $\gamma_{s,y}$  is a constant that depends only on  $s$  and  $y$ , it is real valued for  $s \in \mathbb{R}$ , and it satisfies the uniform bound  $\gamma_{s,y} \ll \log y$  for  $s$  and  $y$  as above.

Finally, we need some control on sums involving  $\Lambda_k = \mu * \log^k$ :

**Lemma 3.6.** *Let  $x, z \geq 3$ ,  $k \in \mathbb{N}$ ,  $m \in \mathbb{N}$ ,  $r \in \mathbb{N} \cup \{0\}$  and  $T \geq 2$ . There exists a constant  $c = c(m)$  such that*

$$\sum_{P^+(n) \leq z} \frac{\Lambda_k(n) \tau_m(n) (\log n)^r}{n^{1+\frac{1}{\log x}}} \leq (c(k+r) \min\{\log z, \log x\})^{k+r}$$

and

$$\int_{-T}^T \left| \sum_{P^+(n) \leq z} \frac{\Lambda_k(n) \tau_m(n) (\log n)^r}{n^{1+\frac{1}{\log x}+it}} \right|^2 dt \leq (c(k+r))^{2k+2r} (T(\log V_T)^{2k+2r} + \log^{2k+2r-1}(\min\{z, x\})).$$

*Proof.* Note that for  $(a, b) = 1$  we have that

$$\begin{aligned}
 \Lambda_k(ab) &= \sum_{d|ab} \mu(d) \log^k(ab/d) = \sum_{\substack{d_1|a \\ d_2|b}} \mu(d_1) \mu(d_2) \log^k\left(\frac{a}{d_1} \cdot \frac{b}{d_2}\right) \\
 (3.2) \quad &= \sum_{j=0}^k \binom{k}{j} \sum_{d_1|a} \sum_{d_2|b} \mu(d_1) \mu(d_2) \log^{k-j}(a/d_1) \log^j(b/d_2) = \sum_{j=0}^k \binom{k}{j} \Lambda_{k-j}(a) \Lambda_j(b).
 \end{aligned}$$

Also, we have that  $\Lambda_k(n) \leq (\log n)^k$  for all  $n \in \mathbb{N}$ , since  $\log^k = 1 * \Lambda_k$ .

Now, every integer  $n$  can be written uniquely as  $n = ab$ , where  $a$  is square-free,  $b$  is square-full and  $(a, b) = 1$ . Togethe with (3.2), this implies that

$$\sum_{P^+(n) \leq z} \frac{\Lambda_k(n) \tau_m(n) (\log n)^r}{n^{1+\frac{1}{\log x}+it}} = \sum_{\substack{0 \leq j_1 \leq k \\ 0 \leq j_2 \leq r}} \binom{k}{j_1} \binom{r}{j_2} \sum_{P^+(a) \leq z} \frac{\mu^2(a) \Lambda_{j_1}(a) \tau_m(a) (\log a)^{j_1}}{a^{1+\frac{1}{\log x}+it}} \cdot C(a),$$

where

$$C(a) = \sum_{\substack{P^+(b) \leq z, (b,a)=1 \\ b \text{ square-full}}} \frac{\Lambda_{k-j_1}(b) \tau_m(b) (\log b)^{r-j_2}}{b^{1+\frac{1}{\log x}+it}}.$$

Since

$$\sum_{\substack{b \leq x \\ b \text{ square-full}}} \tau_m(b) \leq \sum_{g^2 h^3 \leq x} \tau_m(g^2 h^3) \ll_m \sqrt{x} (\log x)^{m-1}$$

and  $\Lambda_{k-j_1}(b) \leq (\log b)^{k-j_1}$ , we find by partial summation that

$$|C(a)| \leq \sum_{b \text{ square-full}} \frac{\tau_m(b) (\log b)^{k+r-j_1-j_2}}{b} \ll_m c_1^{k+r-j_1-j_2} (k+r-j_1-j_2)!$$

for some  $c_1 = c_1(m)$ . The above discussion and Lemma 3.2 show that it suffices to show the lemma with  $\mu^2 \Lambda_k$  in place of  $\Lambda_k$ . In addition, note that if  $P^+(n) \leq z$  and  $\mu^2(n) \Lambda_k(n) \neq 0$ , then  $n \leq z^k$  and  $\tau_m(n) \leq m^k$ , since  $n$  is square-free and it has at most  $k$  distinct prime factors<sup>2</sup>. Therefore  $\tau_m(n)/n^{1+1/\log x} \leq (em)^k/n^{1+1/\log(\min\{x,z\})}$  for all such  $n$ . This reduces the lemma to the case  $z = \infty$  (this is also true for the second part of the lemma, by Lemma 3.2). Finally, we remove the restriction that  $n$  runs over square-free integers, so that the lemma is now reduced to bounding  $(\zeta^{(k)}/\zeta)^{(r)}(1+1/\log x+it)$  pointwise and on average.

We claim that, for any  $s = \sigma + it$  with  $\sigma > 1$  and  $t \in \mathbb{R}$ , we have that

$$(3.3) \quad \left| \left( \frac{\zeta^{(k)}}{\zeta} \right)^{(r)}(s) \right| \leq c_1^{k+r} k! r! \left( \log V_t + \frac{1}{|s-1|} \right)^{k+r},$$

for some absolute constant  $c_1$ . Observe that this estimate immediately implies both parts of the lemma.

So it remains to show (3.3). Lemma 4.1 in [K] implies that

$$(3.4) \quad \left| \left( \frac{\zeta'}{\zeta} \right)^{(r)}(s) \right| \leq c_2^{r+1} r! \left( \log V_t + \frac{1}{|s-1|} \right)^{r+1} \quad (r \in \mathbb{N} \cup \{0\})$$

<sup>2</sup>It is well-known that  $\Lambda_k$  is supported on integers with at most  $k$  distinct prime factors, something which can be seen using the recursive formula  $\Lambda_{k+1} = \Lambda_k \log + \Lambda * \Lambda_k$ .

for some constant  $c_2$ . We will show (3.3) with  $c_1 = 4c_2$  by inducting on  $k$ . When  $k = 1$ , (3.3) holds for all  $r \in \mathbb{N} \cup \{0\}$  by (3.4). Assume now that (3.3) holds for all  $k \in \{1, \dots, K\}$  and all  $r \in \mathbb{N} \cup \{0\}$ . As in the proof of Lemma 3.1, we have that

$$\frac{\zeta^{(K+1)}}{\zeta}(s) = \sum_{j_1=0}^K \binom{K}{j_1} \frac{\zeta^{(K-j_1)}}{\zeta}(s) \left(\frac{\zeta'}{\zeta}\right)^{(j_1)}(s)$$

and, consequently,

$$\left(\frac{\zeta^{(K+1)}}{\zeta}\right)^{(r)}(s) = \sum_{\substack{0 \leq j_1 \leq K \\ 0 \leq j_2 \leq r}} \binom{K}{j_1} \binom{r}{j_2} \left(\frac{\zeta^{(K-j_1)}}{\zeta}\right)^{(r-j_2)}(s) \left(\frac{\zeta'}{\zeta}\right)^{(j_1+j_2)}(s).$$

By the induction hypothesis, (3.4) and the inequality  $(j_1 + j_2)! \leq 2^{j_1+j_2} j_1! j_2!$ , we deduce that

$$\begin{aligned} \left| \left(\frac{\zeta^{(K+1)}}{\zeta}\right)^{(r)}(s) \right| &\leq K! r! \left( \log V_t + \frac{1}{|s-1|} \right)^{K+r+1} \sum_{\substack{0 \leq j_1 \leq K \\ 0 \leq j_2 \leq r}} c_1^{K+r-j_1-j_2} 2^{j_1+j_2} c_2^{j_1+j_2+1} \\ &< c_1^{K+r+1} K! r! \left( \log V_t + \frac{1}{|s-1|} \right)^{K+r+1}, \end{aligned}$$

since  $c_1 = 4c_2$ . This completes the proof of (3.3) and hence of the lemma.  $\square$

#### 4. BOUNDS FOR $L(s, f)$

In this section we estimate the partial sums of  $f$  over integers with no small primes factors and deduce bounds for the derivatives of  $L_y(s, f)$ . Note that the second formula in part (b) of the following Lemma is similar to [Pi76c, Lemma 4].

**Lemma 4.1.** *Let  $f : \mathbb{N} \rightarrow \mathbb{U}$  be a completely multiplicative function such that*

$$\left| \sum_{n \leq x} f(n) \right| \leq \frac{Mx^{\sigma_0}}{(\log x)^A} \quad (x \geq Q)$$

for some  $\sigma_0 \in [3/5, 1]$ ,  $Q \geq 3$ ,  $M \geq 1$  and  $A \geq 0$ . Consider  $x \geq y \geq Q$  with  $\sigma_0 \geq 1 - 1/(60 \log y)$  and  $s = \sigma + it$  with  $\sigma \geq \sigma_1 := (1 + \sigma_0)/2$  and  $t \in \mathbb{R}$ .

(a) *We have that*

$$\sum_{\substack{n \leq x \\ P^-(n) > y}} \frac{f(n)}{n^{it}} \ll \frac{40^A (|t| + 1) (\log y) M x^{\sigma_0}}{(\log x)^A} + \frac{x^{1 - \frac{1}{2 \log y}}}{\log y}.$$

(b) *We have that*

$$\sum_{\substack{n \leq x \\ P^-(n) > y}} \frac{(1 * f)(n)}{n^{it}} = \frac{x^{1-it}}{1-it} L_y(1, f) \prod_{p \leq y} \left( 1 - \frac{1}{p} \right) + O \left( \frac{80^A (|t| + 1) M x^{\sigma_1}}{(\log x)^{A-1}} + \frac{x^{1 - \frac{1}{60 \log y}}}{\log y} \right).$$

Moreover, if  $A > 2$ , then<sup>3</sup>

$$\sum_{\substack{n \leq x \\ P^-(n) > y}} \frac{(1 * f)(n)}{n^s} = \left\{ \left( \frac{1}{s-1} + \gamma_{s,y} \right) L_y(s, f) - \frac{x^{-s+1}}{s-1} L_y(1, f) \right\} \prod_{p \leq y} \left( 1 - \frac{1}{p} \right) + O \left( \frac{A-1}{A-2} \frac{80^A (|t|+1) M}{(\log x)^{A-2}} + x^{-\frac{1}{60 \log y}} \right).$$

*Proof.* (a) We may assume that  $x \geq y^{10}$ ; else, the claimed estimate is trivial. We apply Lemma 3.3 with  $D = x^{1/2}$ :

$$\sum_{\substack{n \leq x \\ P^-(n) > y}} \frac{f(n)}{n^{it}} = \sum_{n \leq x} \frac{f(n)(\lambda^+ * 1)(n)}{n^{it}} + O \left( \sum_{n \leq x} (\lambda^+ * 1 - \lambda^- * 1)(n) \right).$$

Next, we have that

$$\begin{aligned} \sum_{n \leq x} \frac{f(n)(\lambda^+ * 1)(n)}{n^{it}} &= \sum_{\substack{d \leq \sqrt{x} \\ P^+(d) \leq y}} \frac{\lambda^+(d)f(d)}{d^{it}} \sum_{m \leq x/d} \frac{f(m)}{m^{it}} \\ &= \sum_{\substack{d \leq \sqrt{x} \\ P^+(d) \leq y}} \frac{\lambda^+(d)f(d)}{d^{it}} \left( O \left( \left( \frac{x}{d} \right)^{1/20} \right) + \int_{(x/d)^{1/20}}^{x/d} u^{-it} d \left( \sum_{m \leq u} f(m) \right) \right) \\ &\ll x^{0.525} + \frac{40^A (|t|+1) M x^{\sigma_0}}{(\log x)^A} \sum_{P^+(d) \leq y} \frac{1}{d^{\sigma_0}}, \end{aligned}$$

by partial summation. Since  $\max\{3/5, 1 - 1/(60 \log y)\} \leq \sigma_0 \leq 1$ , the formula  $1/p^{\sigma_0} = 1/p + O((1 - \sigma_0)(\log p)/p)$  for  $p \leq y \leq e^{1/(1-\sigma_0)}$  implies that

$$\log \left( \sum_{P^+(d) \leq y} \frac{1}{d^{\sigma_0}} \right) = \log \left( \prod_{p \leq y} \left( 1 - \frac{1}{p^{\sigma_0}} \right)^{-1} \right) = \sum_{p \leq y} \frac{1}{p^{\sigma_0}} + O(1) = \log \log y + O(1).$$

Consequently, we deduce that

$$\sum_{n \leq x} \frac{f(n)(\lambda^+ * 1)(n)}{n^{it}} \ll \frac{40^A M (\log y) x^{\sigma_0}}{(\log x)^A} + x^{0.525},$$

which is admissible. Finally, we have that

$$\sum_{n \leq x} (\lambda^+ * 1 - \lambda^- * 1)(n) = \sum_{\substack{d \leq \sqrt{x} \\ P^+(d) \leq y}} (\lambda^+ - \lambda^-)(d) \left( \frac{x}{d} + O(1) \right) \ll \frac{x^{1 - \frac{1}{2 \log y}}}{\log y},$$

by Lemma 3.3, and part (a) of the lemma follows.

<sup>3</sup>When  $s = 1$ , the right hand side is interpreted to be  $\{L'_y(1, f) + (\log x + \gamma_{1,y})L_y(1, f)\} \prod_{p \leq y} (1 - 1/p) + O(R(x))$  (which agrees with the limit of the right hand side as  $s \rightarrow 1$ ).

(b) Both results are trivial if  $x \leq y^2$ . So assume that  $x > y^2$ . Part (a) and Lemma 3.5 imply that

$$\begin{aligned} \sum_{\substack{n \leq x \\ P^-(n) > y}} \frac{(1 * f)(n)}{n^{it}} &= \sum_{\substack{a \leq \sqrt{x} \\ P^-(n) > y}} \frac{f(a)}{a^{it}} \sum_{\substack{b \leq x/a \\ P^-(a) > y}} \frac{1}{b^{it}} + \sum_{\substack{b \leq \sqrt{x} \\ P^-(b) > y}} \frac{1}{b^{it}} \sum_{\substack{\sqrt{x} < a \leq x/b \\ P^-(n) > y}} \frac{f(a)}{a^{it}} \\ &= \sum_{\substack{a \leq \sqrt{x} \\ P^-(n) > y}} \frac{f(a)}{a^{it}} \frac{(x/a)^{1-it}}{1-it} \prod_{p \leq y} \left(1 - \frac{1}{p}\right) + O\left(\frac{80^A(|t|+1)Mx^{\sigma_1}}{(\log x)^{A-1}} + \frac{x^{1-\frac{1}{60\log y}}}{\log y}\right) \\ &= \frac{x^{1-it}}{1-it} L_y(1, f) \prod_{p \leq y} \left(1 - \frac{1}{p}\right) + O\left(\frac{80^A(|t|+1)Mx^{\sigma_1}}{(\log x)^{A-1}} + \frac{x^{1-\frac{1}{60\log y}}}{\log y}\right), \end{aligned}$$

which proves our first claim.

For the second claim, note that

$$(4.1) \quad \sum_{\substack{n \leq x \\ P^-(n) > y}} \frac{(1 * f)(n)}{n^s} = \sum_{\substack{a \leq \sqrt{x} \\ P^-(n) > y}} \frac{f(a)}{a^s} \sum_{\substack{b \leq x/a \\ P^-(a) > y}} \frac{1}{b^s} + \sum_{\substack{b \leq \sqrt{x} \\ P^-(b) > y}} \frac{1}{b^s} \sum_{\substack{\sqrt{x} < a \leq x/b \\ P^-(n) > y}} \frac{f(a)}{a^s}.$$

Part (a), partial summation and our assumption that  $\sigma \geq \sigma_1 \geq 1 - 1/(120 \log y)$  imply that

$$(4.2) \quad \sum_{\substack{u < a \leq w \\ P^-(a) > y}} \frac{f(a)}{a^s} \ll \frac{40^A \sigma(|t|+1)(\log y)M}{(\log u)^{A-2} u^{\sigma-\sigma_0}} + \sigma u^{1-\sigma-\frac{1}{2\log y}} \quad (w \geq u \geq y).$$

Inserting (4.2) and Lemma 3.5 into (4.1), we find that

$$\begin{aligned} \sum_{\substack{n \leq x \\ P^-(n) > y}} \frac{(1 * f)(n)}{n^s} &= \sum_{\substack{a \leq \sqrt{x} \\ P^-(n) > y}} \frac{f(a)}{a^s} \sum_{\substack{b \leq x/a \\ P^-(a) > y}} \frac{1}{b^s} + O\left(\frac{80^A(|t|+1)M}{(\log x)^{A-2}} + x^{-\frac{1}{5\log y}}\right) \\ &= \sum_{\substack{a \leq \sqrt{x} \\ P^-(n) > y}} \frac{f(a)}{a^s} \left(\frac{1 - (x/a)^{-s+1}}{s-1} + \gamma_{s,y}\right) \prod_{p \leq y} \left(1 - \frac{1}{p}\right) \\ &\quad + O\left(\frac{80^A(|t|+1)M}{(\log x)^{A-2}} + x^{-\frac{1}{60\log y}}\right). \end{aligned}$$

Finally, we extend the summation over  $a$  to all integers with no prime factors  $\leq y$  and we estimate the error term: for  $N > x$  relation (4.2) yields that

$$\begin{aligned} \sum_{\substack{\sqrt{x} < a \leq N \\ P^-(a) > y}} \frac{f(a)}{a^s} \frac{1 - (x/a)^{-s+1}}{s-1} &= \int_1^{\sqrt{x}} \left( \sum_{\sqrt{x} < a \leq x/u} \frac{f(a)}{a^s} \right) \frac{du}{u^s} - \int_0^1 \left( \sum_{x/u < a \leq N} \frac{f(a)}{a^s} \right) \frac{du}{u^s} \\ &\ll \left( \frac{80^A \sigma(|t|+1)(\log y)M}{x^{\frac{\sigma-\sigma_0}{2}} (\log x)^{A-1}} + \frac{\sigma}{x^{\frac{\sigma-1}{2} + \frac{1}{4\log y}}} \right) \int_1^{\sqrt{x}} \frac{du}{u^\sigma} \\ &\quad + \int_0^1 \left( \frac{40^A \sigma(|t|+1)(\log y)M}{(x/u)^{\sigma-\sigma_0} \log^{A-1}(x/u)} + \frac{1}{(x/u)^{\sigma-1+\frac{1}{2\log y}}} \right) \frac{du}{u^\sigma}. \end{aligned}$$

Since

$$\int_1^{\sqrt{x}} \frac{du}{u^\sigma} = \int_0^{\frac{\log x}{2}} e^{v(1-\sigma)} dv \ll \left(1 + x^{\frac{1-\sigma}{2}}\right) \log x,$$

$$\int_0^1 \frac{du}{u^{\sigma_0} \log^{A-1}(N/u)} = \int_0^\infty \frac{dv}{e^{(1-\sigma_0)v} (\log x + v)^{A-1}} \leq \int_{\log x}^\infty \frac{du}{u^{A-1}} = \frac{1}{(A-2)(\log x)^{A-2}}$$

and we have assumed that  $\sigma \geq \sigma_1 \geq 1 - 1/(120 \log y)$ , we deduce that

$$\sum_{\substack{\sqrt{x} < a \leq N \\ P^-(a) > y}} \frac{f(a)}{a^s} \frac{1 - (x/a)^{-s+1}}{s-1} \ll (\log y) \left( \frac{A-1}{A-2} \frac{80^A (|t|+1)M}{(\log x)^{A-2}} + x^{-\frac{1}{60 \log y}} \right)$$

and the lemma follows.  $\square$

Finally, for easy reference, we state the following corollary of Lemma 4.1.

**Lemma 4.2.** *Let  $f : \mathbb{N} \rightarrow \mathbb{C}$  be a completely multiplicative function such that*

$$\left| \sum_{n \leq x} f(n) \right| \leq \frac{x^{\sigma_0} (\log Q)^{A-2}}{(\log x)^A} + \frac{x^{1-1/\log Q}}{(\log x)^2} \quad (x \geq Q)$$

for some  $Q \geq 3$ ,  $A \geq 1$  and  $\sigma_0 \in [1 - 1/(2 \log Q), 1]$ . Consider  $k \in \mathbb{N} \cup \{0\}$ ,  $y \geq Q$  with  $\sigma_0 \geq 1 - 1/(60 \log y)$  and  $\sigma \geq \sigma_1 := (1 + \sigma_0)/2$ .

(a) *We have that*

$$|L_y^{(k)}(\sigma, f)| \ll (ck \log y)^k \min \left\{ \frac{A-k}{A-k-1}, \log \left( 2 + \frac{1}{(\sigma - \sigma_0) \log y} \right) \right\} \quad (0 \leq k \leq A-1).$$

(b) *If  $A \geq 2$  and  $\liminf_{u \rightarrow \infty} \sum_{p \leq u} \Re(f(p))/p = -\infty$ , then for any  $t \in [-1, 1]$  we have that*

$$|L_y^{(k)}(\sigma + it, 1 * f)| \ll (ck \log y)^k \min \left\{ \frac{A-k-1}{A-k-2}, \log \left( 2 + \frac{1}{(\sigma - \sigma_1) \log y} \right) \right\} \quad (0 \leq k \leq A-2).$$

*Proof.* Fix  $k \in \mathbb{Z} \cap [0, A-1]$ . Since

$$e^{u/\log Q} \geq e^{(1-\sigma_0)u} e^{u/(2 \log Q)} \geq \frac{e^{u(1-\sigma_0)} u^k}{k! (2 \log Q)^k} \quad (u \geq 0),$$

we find that

$$\left| \sum_{n \leq x} f(n) \right| \leq \frac{x^{\sigma_0} (\log Q)^{A-2}}{(\log x)^A} + \frac{k! (2 \log Q)^k x^{\sigma_0}}{(\log x)^{k+2}} \leq 2 \cdot \frac{k! x^{\sigma_0} (\log Q)^{\min\{k, A-2\}}}{(\log x)^{\min\{k, A-2\}+2}} \quad (x \geq Q).$$

Part (a) now follows immediately by Lemma 4.1(a) with  $\min\{k+2, A\}$  and  $2 \cdot k! (\log Q)^{\min\{k, A-2\}}$  in place of  $A$  and  $M$ , respectively, and partial summation.

For part (b), note that Lemma 6.1 below implies that

$$\left| L_y \left( 1 + \frac{1}{\log u}, f \right) \right| = \sum_{y < p \leq u} \frac{\Re(f(p))}{p} + O(1) \quad (u > y).$$

So  $L_y(1, f) = \lim_{u \rightarrow \infty} L_y(1 + 1/\log u, f) = 0$ . The claimed result then follows by the first inequality in part (b) of Lemma 4.1 and partial summation.  $\square$

5. BOUNDS FOR  $\frac{1}{L}(s, f)$  AND  $\frac{L'}{L}(s, f)$ 

In this section we list some estimates for the derivatives of  $\frac{1}{L}(s, f)$  and  $\frac{L'}{L}(s, f)$ , which we shall need in the proof of Theorems 1.2 and 2.1. The next two lemmas use an idea from [IK, p. 40-42], also exploited in [K].

**Lemma 5.1.** *Let  $s = \sigma + it$  with  $\sigma > 1$  and  $t \in \mathbb{R}$ ,  $k \in \mathbb{N}$ ,  $Q \geq 2$  and  $M > 0$ . Consider a multiplicative function  $f : \mathbb{N} \rightarrow \mathbb{U}$  such that*

$$|L_Q^{(j)}(s, f)| \leq j!M^j \quad (1 \leq j \leq k).$$

*There is an absolute constant  $c$  such that for  $z \geq 3/2$  we have that*

$$\left| \left( \frac{L'_z}{L_z} \right)^{(k-1)}(s, f) \right| \ll c^k k! \left( \frac{M}{\min\{1, |L_Q(s, f)|\}} + \log(zQ) \right)^k.$$

*Proof.* Note that

$$\left( \frac{-L'_z}{L_z} \right)^{(k-1)}(s, f) = \left( \frac{-L'_Q}{L_Q} \right)^{(k-1)}(s, g) + O(c_1 k \log(zQ))^k$$

and

$$\frac{1}{j!} \left| \frac{L_Q^{(j)}}{L_Q}(s, f) \right| \leq \frac{M^j}{|L_Q(\sigma, f)|} \leq \left( \frac{M}{\min\{|L_Q(s, f)|, 1\}} \right)^j \quad (1 \leq j \leq k).$$

So Lemma 3.1 applied with  $F(s) = L_Q(s, f)$  implies that

$$(5.1) \quad \left| \left( \frac{L'_Q}{L_Q} \right)^{(k-1)}(s) \right| \leq k! \left( \frac{8M}{\min\{1, |L_Q(s, f)|\}} \right)^k,$$

which completes the proof.  $\square$

**Lemma 5.2.** *Let  $s = \sigma + it$  with  $\sigma > 1$  and  $t \in \mathbb{R}$ ,  $k \in \mathbb{N} \cup \{0\}$ ,  $Q \geq 2$ ,  $M \geq 1$  and  $c \geq 1$ . Consider a multiplicative function  $f : \mathbb{N} \rightarrow \mathbb{U}$  such that*

$$|L_Q(s, 1 * f)| \leq c \quad \text{and} \quad |L_Q^{(j)}(s, 1 * f)| \leq j!M^j \quad (1 \leq j \leq k).$$

*Then there is an absolute constant  $c_0$  such that*

$$(5.2) \quad |L_z^{(k)}(s, f)| \leq c_0^k k! \frac{\log(V_t Q) (M^k + c \log^k(V_t Q))}{\log z}$$

*for all  $z \in [3/2, Q]$ . Also, if  $k \geq 1$ , then*

$$(5.3) \quad \left| \left( \frac{L'_2}{L_2} \right)^{(k-1)}(s, f) + \frac{(-1)^k (k-1)!}{(s-1)^k} \right| \leq (c_0 k)^k \left( \frac{M}{\min\{1, |L_Q(s, 1 * f)|\}} + \log(QV_t) \right)^k.$$

*Proof.* Set  $y = \max\{Q, V_t^{100}\}$ . We have that

$$L_z(s, f) = L_Q(s, 1 * f) \frac{1}{\zeta_y(s)} \sum_{p|n \Rightarrow z < n \leq y} \frac{g(n)}{n^s},$$



where  $g(n_1 n_2) = f(n_1) \mu(n_2)$  for  $n_1 \in \{m \in \mathbb{N} : p|m \Rightarrow z < p \leq Q\}$  and  $n_2 \in \{m \in \mathbb{N} : p|m \Rightarrow Q < p \leq y\}$ . Consequently, we find that

$$(5.4) \quad L_z^{(k)}(s, f) = \sum_{k_1+k_2+k_3=k} \frac{k!}{k_1!k_2!k_3!} L_Q^{(k_1)}(s, 1 * f) \left( \frac{1}{\zeta_y} \right)^{(k_2)}(s) \sum_{p|n \Rightarrow z < p \leq y} \frac{g(n)(-\log n)^{k_3}}{n^s}.$$

In order to continue, we need to bound the derivatives of  $1/\zeta_y(s)$ . We claim that

$$(5.5) \quad \left( \frac{1}{\zeta_y} \right)^{(m)}(s) \ll m! (c_1 \log y)^m \quad (m \geq 0).$$

This estimate, together with Lemma 3.4 and our assumption on the derivatives of  $L_Q(s, 1 * f)$  into (5.4), immediately implies (5.2).

In order to show (5.5), let  $C \geq 1$  be such that  $|\gamma_{s,y}| \leq C \log y$ , where  $\gamma_{s,y}$  is defined by Lemma 3.5. Note that

$$(5.6) \quad \zeta_y^{(m)}(s) \ll m! (c_2 \log y)^m \quad (m \geq 0),$$

by the first formula in Lemma 3.5 and partial summation. If, in addition,  $|s-1| \geq 1/(2C \log y)$ , then

$$(5.7) \quad |\zeta_y(s)| \asymp 1,$$

trivially if  $\sigma \geq 1 + 1/(4C \log y)$  and by [K, Lemma 4.2]<sup>4</sup> if  $|t| \geq 1/(4C \log y)$ . So in this case (5.5) follows by Lemma 3.1. Finally, assume that  $|s-1| \leq 1/(2C \log y)$ . If we set

$$F(\tau) = (\tau-1) \zeta_y(\tau) \prod_{p \leq y} \left( 1 - \frac{1}{p} \right)^{-1},$$

then

$$F^{(j)}(s) = ((s-1) \zeta_y^{(j)}(s) + j \zeta_y^{(j-1)}(s)) \prod_{p \leq y} \left( 1 - \frac{1}{p} \right)^{-1} \ll j! (c_3 \log y)^j \quad (j \geq 1),$$

by (5.6). Moreover, we have that

$$(5.8) \quad |F(s)| \asymp 1$$

by Lemma 3.5. So Lemma 3.1 implies that

$$\left( \frac{1}{F} \right)^{(r)}(s) \ll r! (c_4 \log y)^r \quad (r \geq 0)$$

and, consequently,

$$\left( \frac{1}{\zeta_y} \right)^{(m)}(s) = \left\{ (s-1) \left( \frac{1}{F} \right)^{(m)}(s) + m \left( \frac{1}{F} \right)^{(m-1)}(s) \right\} \prod_{p \leq y} \left( 1 - \frac{1}{p} \right)^{-1} \ll (c_5 m \log y)^m,$$

which completes the proof of (5.5) in this last case too and hence of (5.2).

Finally, in order to show (5.3), note that

$$\left| \left( \frac{L'_Q}{L_Q} \right)^{(k-1)}(s, 1 * f) \right| \ll k! \left( \frac{c_4 M}{\min\{1, |L_Q(s, 1 * f)|\}} + \log Q \right)^k,$$

---

<sup>4</sup>This lemma holds only when  $|t| \geq 1/\log y$ , but its proof can be immediately extended to the case  $|t| \geq 1/(4C \log y)$ .

by Lemma 5.1. Combining this estimate with Lemma 4.3 in [K] and the formula

$$\left(\frac{L'_2}{L_2}\right)^{(k-1)}(s, f) = \left(\frac{L'_Q}{L_Q}\right)^{(k-1)}(s, 1 * f) - \left(\frac{\zeta'}{\zeta}\right)^{(k-1)}(s) + O((c_6 k \log Q)^k)$$

yields (5.3), thus completing the proof of the lemma.  $\square$

## 6. DISTANCES OF MULTIPLICATIVE FUNCTIONS

This section is devoted to studying some properties of the distance function and establishing Theorems 2.2, 2.3 and 2.4. We start with two straightforward results, which we state for easy reference.

**Lemma 6.1.** *Let  $x, y \geq 2$ ,  $t \in \mathbb{R}$  and  $f : \mathbb{N} \rightarrow \mathbb{U}$  a multiplicative function. Then*

$$\log \left\{ L_y \left( 1 + \frac{1}{\log x} + it, f \right) \right\} = \sum_{y < p \leq x} \frac{f(p)}{p^{1+it}} + O(1).$$

*Proof.* The lemma follows by writing  $L_y(s, f)$  as an Euler product and then performing a standard calculation.  $\square$

**Lemma 6.2.** *Let  $f : \mathbb{N} \rightarrow \mathbb{U}$  be a completely multiplicative function,  $\epsilon > 0$  and  $Q \geq 3$  such that*

$$\left| \sum_{n \leq x} f(n) \right| \leq \frac{x}{(\log Q)^{1-\epsilon} (\log x)^{1+\epsilon}} \quad (x \geq Q).$$

*There is a positive constant  $c = c(\epsilon)$  such that for every  $x \geq y \geq Q$  we have that*

$$\sum_{y < p \leq x} \frac{\Re(f(p))}{p} \leq c.$$

*Proof.* The lemma follows immediately by Lemmas 4.2(a) and 6.1.  $\square$

Next, we have the triangle inequality [GS] for the distance function defined in Section 1.

**Lemma 6.3.** *Let  $f, g, h : \mathbb{N} \rightarrow \mathbb{U}$  be multiplicative functions and  $x \geq y \geq 1$ . Then*

$$\mathbb{D}(f(n), g(n); y, x) + \mathbb{D}(g(n), h(n); y, x) \geq \mathbb{D}(f(n), h(n); y, x).$$

The following result is Lemma 3.3 in [K].

**Lemma 6.4.** *Let  $y_2 \geq y_1 \geq y_0 \geq 2$ . Consider a multiplicative function  $f : \mathbb{N} \rightarrow \mathbb{U}$  such that*

$$\left| L'_{y_0} \left( 1 + \frac{1}{\log x}, f \right) \right| \leq c \log y_0 \quad (y_1 \leq x \leq y_2)$$

*for some  $c \geq 1$  and*

$$\mathbb{D}^2(f(n), \mu(n); y_0, x) \geq \delta \log \left( \frac{\log x}{\log y_0} \right) - M \quad (y_1 \leq x \leq y_2)$$

*for some  $\delta > 0$  and  $M \geq 0$ . Then*

$$\left| \sum_{y_1 < p \leq y_2} \frac{f(p)}{p} \right| \ll_{c, M} \frac{1}{\delta}.$$

We are now in position to show Theorem 2.4.

*Proof of Theorem 2.4.* (a) Assume that  $f^2$  satisfies (2.6). Then Lemmas 6.2 and 6.3 imply that

$$\begin{aligned} 4 \cdot \mathbb{D}^2(f(n), \mu(n); y, x) &= \left( \mathbb{D}(f(n), \mu(n); y, x) + \mathbb{D}(\mu(n), \overline{f(n)}; y, x) \right)^2 \\ &\geq \mathbb{D}^2(f(n), \overline{f(n)}; y, x) = \mathbb{D}^2(f^2(n), 1; y, x) \geq \log \left( \frac{\log x}{\log y} \right) + O(1) \end{aligned}$$

for  $x \geq y \geq Q$ . So part (a) of the theorem follows by Lemma 6.4.

(b) Assume that  $f(p) \in \mathbb{R}$  for  $p \geq Q$  and consider  $t \in \mathbb{R}$  with  $Q \geq \max \{V_t, e^{1/|t|}\}$ . Then Lemmas 6.1, 6.3 and 3.5 imply that

$$\begin{aligned} 4 \cdot \mathbb{D}^2(f(n), \mu(n)n^{it}; y, x) &= \left( \mathbb{D}(\mu(n)n^{-it}, f(n); y, x) + \mathbb{D}(f(n), \mu(n)n^{it}; y, x) \right)^2 \\ &\geq \mathbb{D}^2(\mu(n)n^{-it}, \mu(n)n^{it}; y, x) = \mathbb{D}^2(1, n^{2it}; y, x) \\ &= \log \left( \frac{\log x}{\log y} \right) + O(1) - \log \left| \zeta_y \left( 1 + \frac{1}{\log x} + 2it \right) \right| \\ &\geq \log \left( \frac{\log x}{\log y} \right) + O(1) \end{aligned}$$

for  $x \geq y \geq Q$ . So Lemma 6.4 yields that

$$(6.1) \quad \sum_{y < p \leq x} \frac{f(p)}{p^{1+it}} \ll 1 \quad (x \geq y \geq Q),$$

which completes the proof in this case.

Finally, assume that  $|t| \leq 1/\log Q$ . Let  $x \geq Q$  and  $z = \min\{x, e^{1/|t|}\} \geq Q$ . Then

$$\sum_{Q < p \leq x} \frac{\Re(f(p)p^{-it})}{p} = \sum_{Q < p \leq z} \frac{f(p) + O(|t| \log p)}{p} + \sum_{z < p \leq x} \frac{\Re(f(p)p^{-it})}{p} = \sum_{Q < p \leq z} \frac{f(p)}{p} + O(1),$$

by (6.1). So for every  $u \geq x$  Lemmas 6.1 and 6.2 yield that

$$\sum_{Q < p \leq x} \frac{\Re(f(p)p^{-it})}{p} \geq \sum_{Q < p \leq x} \frac{f(p)}{p} + O(1) \geq \sum_{Q < p \leq u} \frac{f(p)}{p} + O(1) = \log L_Q \left( 1 + \frac{1}{\log u}, f \right) + O(1).$$

Letting  $u \rightarrow \infty$  completes the proof of the theorem in this last case too.  $\square$

Before we show Theorem 2.2, we need a preliminary result, which strengthens Lemma 6.4.

**Lemma 6.5.** *Let  $y_1 \geq y_0 \geq 2$  and let  $f : \mathbb{N} \rightarrow \mathbb{U}$  be a multiplicative function such that*

$$\left| L'_{y_0} \left( 1 + \frac{1}{\log x}, f \right) \right| \leq c_1 \log y_0 \quad (x \geq y_0)$$

and

$$\mathbb{D}^2(f(n), \mu(n); y_0, x) \geq \delta \log \left( \frac{\log x}{\log y_0} \right) - M \quad (y_0 \leq x \leq y_1)$$

for some  $c_1 > 0$ ,  $\delta > 0$  and  $M \geq 0$ . There is a constant  $c_2$ , depending at most on  $c_1$ , such that if  $y_1 \geq y_0^{\exp\{c_2 M/\delta\}}$ , then

$$\sum_{y_1 < p \leq z} \frac{f(p)}{p} \ll_{c_1} \frac{1}{\delta} \quad (z \geq y_1).$$

*Proof.* Let  $y_1 \geq y_0^{\exp\{c_2 M/\delta\}}$ . Set  $y'_0 = y_0^{\exp\{2M/\delta\}}$  and note that

$$(6.2) \quad \mathbb{D}^2(f(n), \mu(n); y_0, x) \geq \frac{\delta}{2} \log \left( \frac{\log x}{\log y_0} \right) \geq c_3 \delta \sum_{y_0 < p \leq x} \frac{1}{p} \quad (y'_0 \leq x \leq y_1)$$

for some constant  $c_3 \leq 1/2$ . For  $y \geq y_1 \geq y'_0$  set

$$\epsilon(y) = \min_{y'_0 \leq x \leq y} \frac{\mathbb{D}^2(f(n), \mu(n); y_0, x)}{\sum_{y_0 < p \leq x} 1/p}$$

and note that

$$(6.3) \quad \left| \sum_{y'_0 < p \leq x} \frac{f(p)}{p} \right| \ll_{c_1} \frac{1}{\epsilon(y)} \quad (y'_0 \leq x \leq y),$$

by Lemma 6.4. We claim that

$$(6.4) \quad \epsilon(y) \geq c_3 \delta.$$

Assume on the contrary that  $\epsilon(y) < c_3 \delta$ . Let  $x_0 \in [y'_0, y]$  be such that

$$\mathbb{D}^2(f(n), \mu(n); y_0, x_0) = \epsilon(y) \sum_{y_0 < p \leq x_0} \frac{1}{p}.$$

We must have that  $x_0 > y_1$ ; else, (6.2) would contradict our assumption that  $\epsilon(y) < c_3 \delta$ . Moreover, we have that

$$(6.5) \quad \begin{aligned} \mathbb{D}^2(f(n), \mu(n); y_0, x_0) &= \epsilon(y) \sum_{y_0 < p \leq x_0} \frac{1}{p} = \frac{\epsilon(y)}{1 - \epsilon(y)} \sum_{y_0 < p \leq x_0} \frac{-\Re(f(p))}{p} \\ &= O\left(\frac{\epsilon(y)M}{\delta}\right) + \frac{\epsilon(y)}{1 - \epsilon(y)} \sum_{y'_0 < p \leq x_0} \frac{-\Re(f(p))}{p} \ll_{c_1} M, \end{aligned}$$

by (6.3) and our assumption that  $\epsilon(y) < c_3 \delta$ . On the other hand, we have that

$$\mathbb{D}^2(f(n), \mu(n); y_0, x_0) \geq \mathbb{D}^2(f(n), \mu(n); y_0, y_1) \geq \frac{\delta}{2} \log \left( \frac{\log y_1}{\log y_0} \right) \geq \frac{c_2 M}{2},$$

by (6.2). If  $c_2$  is large enough, the above inequality contradicts (6.5). This implies that relation (6.4) does indeed hold. Combining relations (6.3) and (6.4), we deduce that

$$\sum_{y'_0 < p \leq x} \frac{f(p)}{p} \ll_{c_1} \frac{1}{\delta} \quad (y'_0 \leq x \leq y).$$

Since the above inequality is true for all  $y \geq y_1$ , the desired result follows.  $\square$

*Proof of Theorem 2.2.* Note that if we prove the existence of  $Q'$  with the desired properties, it follows immediately that  $|L_y(1, f)| \asymp (\log y)/\log(yQ')$ , by Lemma 6.1.

Let  $c$  be a large constant, depending at most on  $\epsilon$ , to be chosen later, and assume, without loss of generality, that  $Q \geq c$ . Define  $Y$  to be the smallest integer  $y \geq Q$  such that

$$(6.6) \quad \mathbb{D}^2(f(n), \mu(n); y, y^c) \leq \frac{1}{\log^2 c},$$

if such an integer exists; else, set  $Y = \infty$ . This definition immediately implies that

$$(6.7) \quad \mathbb{D}^2(f(n), \mu(n); Q, z) \geq \frac{c_1}{\log^3 c} \log \left( \frac{\log z}{\log Q} \right) - O(\log c)$$

for  $Q \leq z < Y$ , where  $c_1$  is some appropriate absolute constant. By the above relation and Lemmas 4.2(a) and 6.5, there is a constant  $c_2 = c_2(\epsilon)$ , independent of our choice of  $c$ , such that if  $Y \geq Q_1 = Q^{\exp\{c_2 \log^4 c\}}$ , then

$$\sum_{Q_1 < p \leq z} \frac{f(p)}{p} \ll_{\epsilon} \log^3 c \quad (z > Q_1).$$

So the theorem follows in this case by taking  $Q' = Q_1 = Q^{O_c(1)}$ . Consequently, we may assume that  $Y < Q_1$ . Consider  $y \geq Q$  that satisfies (6.6). We have that

$$\sum_{y < p \leq y^c} \frac{|1 + f(p)|}{p} \leq \sum_{y < p \leq y^c} \frac{\sqrt{2(1 + \Re(f(p)))}}{p} \leq \mathbb{D}(f(n), \mu(n); y, y^c) \left( \sum_{y < p \leq y^c} \frac{2}{p} \right)^{1/2} \ll \frac{1}{\sqrt{\log c}},$$

by the Cauchy-Schwarz inequality and the inequality  $|1 + u|^2 \leq 2\Re(1 + u)$  for  $u \in \mathbb{U}$ . Therefore we have that

$$\begin{aligned} \sum_{\substack{1 < n \leq y^c \\ P^-(n) > y}} \frac{|1 * f|(n)}{n} &\leq -1 + \prod_{y < p \leq y^c} \left( 1 + \frac{|1 * f|(p)}{p} + \frac{|1 * f|(p^2)}{p^2} + \dots \right) \\ &= -1 + \exp \left\{ \sum_{y < p \leq y^c} \frac{|1 + f(p)|}{p} + O\left(\frac{1}{y}\right) \right\} \ll \frac{1}{\sqrt{\log c}}, \end{aligned}$$

since  $y \geq Q \geq c$ . On the other hand, Lemma 4.1(b) yields that

$$\sum_{\substack{n \leq y^c \\ P^-(n) > y}} \frac{(1 * f)(n)}{n^\sigma} = \left\{ \left( \frac{1}{\sigma - 1} + \gamma_{\sigma, y} \right) L_y(\sigma, f) - \frac{y^{-c(\sigma-1)}}{\sigma - 1} L_y(1, f) \right\} \prod_{p \leq y} \left( 1 - \frac{1}{p} \right) + O_\epsilon(c^{-\epsilon})$$

uniformly for  $\sigma > 1$ . So we deduce that

$$(6.8) \quad \left\{ \left( \frac{1}{\sigma - 1} + \gamma_{\sigma, y} \right) L_y(\sigma, f) - \frac{y^{-c(\sigma-1)}}{\sigma - 1} L_y(1, f) \right\} \prod_{p \leq y} \left( 1 - \frac{1}{p} \right) = 1 + O_\epsilon \left( \frac{1}{\sqrt{\log c}} \right).$$

If  $L_Q(1, f) = 0$ , then  $L_y(1, f) = 0$ . Moreover, letting  $x \rightarrow \infty$  in the second formula in Lemma 3.5 yields the identity

$$\zeta_y(\sigma) = \left( \frac{1}{\sigma - 1} + \gamma_{\sigma, y} \right) \prod_{p \leq y} \left( 1 - \frac{1}{p} \right).$$

So (6.8) becomes

$$\zeta_y(\sigma) L_y(\sigma, f) = 1 + O_\epsilon \left( \frac{1}{\sqrt{\log c}} \right).$$

If we choose  $c = c(\epsilon)$  large enough and we set  $\sigma = 1 + 1/\log x$  for some  $x \geq y$  in the above formula, then Lemma 6.1 implies that  $\mathbb{D}^2(f(n), \mu(n); y, x) \ll 1$ . Selecting  $y = Y = Q^{O_c(1)}$  then completes the proof of the theorem in this case by taking  $Q' = \infty$ .

Lastly, we consider the case  $L_Q(1, f) \neq 0$ . Letting  $\sigma \rightarrow 1^+$  in (6.8) and dividing the resulting formula by

$$P(y) = L_y(1, f) \prod_{p \leq y} \left(1 - \frac{1}{p}\right) = L(1, f) \prod_{p \leq y} \left(1 - \frac{f(p)}{p}\right) \left(1 - \frac{1}{p}\right),$$

gives us that

$$(6.9) \quad c \log y + \gamma_{1,y} - \frac{L'_y}{L_y}(1, f) = \frac{1 + O_\epsilon(\log^{-1/2} c)}{P(y)}.$$

Since  $\gamma_{1,y} \ll \log y$  by Lemma 3.5 and  $\frac{L'_y}{L_y}(1, f) = \frac{L'}{L}(1, f) + O(\log y)$ , relation (6.9) becomes

$$(6.10) \quad c \log y = \frac{L'}{L}(1, f) + \frac{1 + O_\epsilon(\log^{-1/2} c)}{P(y)} + O(\log y).$$

Consider now two numbers  $y_2 \geq y_1^2$  which satisfy (6.6). Then (6.10) is true for  $y = y_1$  and  $y = y_2$  and subtracting the first one of these formulas from the second one yields the estimate

$$(6.11) \quad \frac{1}{P(y_2)} - \frac{1}{P(y_1)} + O_\epsilon \left( \frac{1}{\sqrt{\log c}} \left( \frac{1}{P(y_1)} + \frac{1}{P(y_2)} \right) \right) = c \log \frac{y_2}{y_1} + O(\log y_2).$$

Note that

$$\left| \frac{P(y_1)}{P(y_2)} \right| = \exp \left\{ O \left( \frac{1}{y_1} \right) + \mathbb{D}^2(f(n), \mu(n); y_1, y_2) \right\}.$$

Therefore, if  $\mathbb{D}(f(n), \mu(n); y_1, y_2) \geq 1$  and  $c$  is large enough, then (6.11) yields that

$$\frac{1}{|P(y_2)|} \asymp c \log y_2,$$

which implies that  $|L_{y_2}(1, f)| \asymp 1/c$ . Combining this with (6.6) for  $y = y_2$ , we find that

$$(6.12) \quad |L_{y_2^c}(f, 1)| \asymp |L_{y_2}(1, f)| \exp \left\{ - \sum_{y_2 < p \leq y_2^c} \frac{\Re(f(p))}{p} \right\} \asymp \frac{1}{c} \exp\{\log c\} = 1.$$

Having proven this, it is relatively easy to complete the proof of the theorem. First, define a sequence  $Y_1, Y_2, \dots$  inductively as follows. Set  $Y_1 = Y$  and let  $Y_{j+1}$  be the smallest integer  $y \geq Y_j^c$  which satisfies (6.6). Since  $L_Q(1, f) \neq 0$ , this sequence is finite, say of  $J$  elements. If  $J \leq 2$ , then (6.7) holds for all  $z \geq Q$ , possibly with a different implied constant in the term  $O(\log c)$ , and Lemmas 4.2(a) and 6.4 complete the proof of the theorem with  $Q' = Q$ . So assume that  $J > 2$ . Then for  $u > Y_J^c$  we have that

$$\begin{aligned} \left| L_{Y_{J-1}^c} \left( 1 + \frac{1}{\log u}, f \right) \right| &\asymp \left| L_{Y_{J-1}^c} \left( 1 + \frac{1}{\log Y_J} \right) L_{Y_J^c} \left( 1 + \frac{1}{\log u}, f \right) \right| \exp \left\{ \sum_{Y_J < p \leq Y_J^c} \frac{\Re(f(p))}{p} \right\} \\ &\ll \exp \left\{ \sum_{Y_J < p \leq Y_J^c} \frac{\Re(f(p))}{p} \right\} \ll \frac{1}{c}, \end{aligned}$$

by Lemmas 6.1 and 4.2(a) and by (6.6) with  $y = Y_J$ . Consequently,

$$(6.13) \quad |L_{Y_{J-1}^c}(1, f)| = \lim_{u \rightarrow \infty} \left| L_{Y_{J-1}^c} \left( 1 + \frac{1}{\log u}, f \right) \right| \ll \frac{1}{c}.$$

This implies that

$$(6.14) \quad \mathbb{D}(f(n), \mu(n); Y_1, Y_{J-1}) < 1;$$

else, (6.12) with  $y_2 = Y_{J-1}$  would yield the estimate  $|L_{Y_{J-1}^c}(1, f)| \asymp 1$ , which contradicts (6.13), provided that  $c$  is large enough. Also, we have that

$$\mathbb{D}^2(f(n), \mu(n); Y_{J-1}, z) \geq \frac{c_3}{\log^3 c} \log \left( \frac{\log z}{\log Y_{J-1}} \right) - O(\log c) \quad (z \geq Y_{J-1}),$$

by the definition of  $Y_{J-1}$ , which together with Lemmas 4.2(a) and 6.4 implies that

$$(6.15) \quad \sum_{Y_{J-1} < p \leq z} \frac{f(p)}{p} \ll_c 1 \quad (z \geq Y_{J-1}).$$

So the theorem follows in this last case too with  $Q' = Y_{J-1}$ .  $\square$

Finally, we show Theorem 2.3.

*Proof of Theorem 2.3.* It suffices to show the theorem when  $Q$  is large enough. Note that  $\eta \ll 1$ , by Lemma 4.2(a). Set  $X = e^{1/(\sigma-1)}$ . If  $X < Q$ , then  $|L_Q(\sigma + it, f)| \asymp 1$  for all  $t \in [-\tau, \tau]$ , by Lemma 6.1, and the lemma follows. So for the rest of the proof we assume that  $X > Q$ . For each  $t \in [-\tau, \tau]$  Theorem 2.4 implies that there is some  $C'_t \in [Q, +\infty]$  such that

$$(6.16) \quad \mathbb{D}(f(n), \mu(n)n^{-it}; Q, C'_t) \ll_\epsilon 1 \quad \text{and} \quad \sum_{C'_t < p \leq z} \frac{f(p)}{p^{1+it}} \ll_\epsilon 1 \quad (z > C'_t, |t| \leq \tau).$$

In particular,  $|L_Q(\sigma + it, f)| \asymp_\epsilon \log Q / \min\{\log X, \log C'_t\}$  by Lemma 6.1. So if we set  $C_t = \min\{C'_t, X\} \geq Q$ , then  $\eta \asymp_\epsilon (\log Q) / \log C_{t_0}$ , so that the theorem is equivalent to

$$(6.17) \quad |L_Q(\sigma + it, f)| \asymp_\epsilon \begin{cases} (\log Q) / \log C_{t_0} & \text{if } |t - t_0| \leq 1 / \log C_{t_0}, \\ |t - t_0| \log Q & \text{if } 1 / \log C_{t_0} \leq |t - t_0| \leq 1 / \log Q, \\ \log Q & \text{if } |t - t_0| \geq 1 / \log Q. \end{cases}$$

First, note that  $C_{t_0} \geq C_t^{-O_\epsilon(1)}$  for every  $t \in J$ , by the choice of  $t_0$ . Thus if  $|t - t_0| \leq 1 / \log C_{t_0}$ , then relation (6.16) and the formula  $p^{i(t-t_0)} = 1 + O(|t - t_0| \log p)$ , which is valid for  $p \leq C_{t_0} \leq e^{1/|t-t_0|}$ , imply that

$$\begin{aligned} \sum_{Q < p \leq X} \frac{\Re(f(p)p^{-it})}{p} &= O_\epsilon(1) + \sum_{Q < p \leq C_{t_0}} \frac{\Re(f(p)p^{-it})}{p} = O_\epsilon(1) + \sum_{Q < p \leq C_{t_0}} \frac{\Re(f(p)p^{-it_0})}{p} \\ &= O_\epsilon(1) - \sum_{Q < p \leq C_{t_0}} \frac{1}{p} = O_\epsilon(1) - \log \left( \frac{\log C_{t_0}}{\log Q} \right). \end{aligned}$$

So Lemma 6.1 completes the proof of (6.17) in this case.

Fix now  $t \in J$  with  $|t - t_0| \geq 1 / \log C_{t_0}$  and consider  $y \in [\max\{Q, e^{1/|t-t_0|}\}, C_{t_0}]$ . For  $z \in [y, C_{t_0}]$  we have that

$$\sum_{y < p \leq z} \frac{|1 + f(p)p^{-it_0}|}{p} \leq \mathbb{D}(f(n), \mu(n)n^{it_0}; y, z) \left( \sum_{y < p \leq z} \frac{2}{p} \right)^{1/2} \ll \log^{1/2} \left( \frac{2 \log z}{\log y} \right),$$

by (6.16), the Cauchy-Schwarz inequality and the inequality  $|1 + u|^2 \leq 2\Re(1 + u)$  for  $u \in \mathbb{U}$ . Consequently, we deduce that

$$\begin{aligned} \mathbb{D}^2(f(n), \mu(n)n^{it}; y, z) &= O\left(\log^{1/2}\left(\frac{\log z}{\log y}\right)\right) + \mathbb{D}^2(1, n^{i(t-t_0)}; y, z) \\ &\geq \log\left(\frac{\log z}{\log y}\right) + O\left(\log^{1/2}\left(\frac{2 \log z}{\log y}\right)\right) \end{aligned}$$

for  $y \leq z \leq C_{t_0}$ , by Lemmas 6.1 and 3.5, since  $y \geq Q \geq (V_{t-t_0})^{100}$  by our assumption on the size of  $\tau$ . So Lemmas 6.4 and 4.2(a) imply that

$$\sum_{y < p \leq C_{t_0}} \frac{f(p)}{p^{1+it}} \ll 1 \quad (\max\{Q, e^{1/|t-t_0|}\} \leq y \leq C_{t_0}).$$

The above estimate, (6.16) and the fact that  $C_{t_0} \geq C_t^{-O_\epsilon(1)}$  imply that

$$(6.18) \quad \sum_{y < p \leq X} \frac{f(p)}{p^{1+it}} \ll 1 \quad (\max\{Q, e^{1/|t-t_0|}\} \leq y \leq X).$$

Since we have assumed that  $e^{1/|t-t_0|} \leq C_{t_0} \leq X$ , relation (6.18) and the formula  $\Re(f(p)p^{-it}) = \Re(f(p)p^{-it_0}) + O(|t-t_0| \log p)$ , which is valid for  $p \leq e^{1/|t-t_0|}$ , imply that

$$\sum_{Q < p \leq X} \frac{\Re(f(p)p^{-it})}{p} = O_\epsilon(1) + \sum_{Q < p \leq e^{1/|t-t_0|}} \frac{\Re(f(p)p^{-it})}{p} = O_\epsilon(1) + \sum_{Q < p \leq e^{1/|t-t_0|}} \frac{\Re(f(p)p^{-it_0})}{p}.$$

Therefore relation (6.16) and our assumption that  $e^{1/|t-t_0|} \leq C_{t_0}$  imply that

$$\sum_{Q < p \leq X} \frac{\Re(f(p)p^{-it})}{p} = O_\epsilon(1) - \sum_{Q < p \leq e^{1/|t-t_0|}} \frac{1}{p} = O_\epsilon(1) - \log\left(\frac{\log(\max\{Q, e^{1/|t-t_0|}\})}{\log Q}\right).$$

Together with Lemma 6.1, this implies relation (6.17) in this last case too, thus completing the proof of the theorem.  $\square$

## 7. REAL ZEROES AND THE SIZE OF $L(1, f)$

In this section we prove Theorem 2.5.

*Proof of Theorem 2.5.* For  $\sigma > 1 - 1/\log Q$  and  $y \geq Q$  we have that

$$\begin{aligned} (7.1) \quad L_y(\sigma, f) &= \lim_{N \rightarrow \infty} \sum_{n \leq N} \frac{f(n)}{n^\sigma} \sum_{\substack{d|n \\ P^+(d) \leq y}} \mu(d) = \sum_{P^+(d) \leq y} \frac{\mu(d)f(d)}{d^\sigma} \lim_{N \rightarrow \infty} \sum_{n \leq N} \frac{f(n)}{n^\sigma} \\ &= L(\sigma, f) \prod_{p \leq y} \left(1 - \frac{f(p)}{p^\sigma}\right), \end{aligned}$$

by our assumption that  $f$  is totally multiplicative. Thus for  $y \geq Q$

$$(7.2) \quad L_y(\sigma, f) = L_Q(\sigma, f) \prod_{Q < p \leq y} \left(1 - \frac{f(p)}{p^\sigma}\right).$$



Next, by Lemma 3.5, there is a constant  $M \geq 120$  such that  $\gamma_{1-\eta,y} \in [-M \log y, M \log y]$  for all  $y \geq 3$  and all  $\eta \in [0, 1/(60 \log y)]$ . We claim that for  $0 \leq \eta \leq 1/(M \log Q)$  we have the relation

$$(7.3) \quad L_Q(1 - \eta, f) \geq 0 \quad \implies \quad L_Q(1, f) \gg \eta \log Q.$$

Indeed, set  $y = e^{1/(M\eta)} \geq Q$ , so that  $\gamma_{1-\eta,y} \leq M \log y = 1/\eta$ . Since  $f(p) \in \mathbb{R}$  for  $p > Q$ , if  $L_Q(1 - \eta, f) \geq 0$ , then relation (7.2) yields that  $L_y(1 - \eta, f) \geq 0$ . Thus Lemma 4.1(b) with  $A = 3$ ,  $\sigma_0 = 1 - 1/(2 \log Q)$ ,  $M \asymp \log y$ ,  $\sigma = 1 - \eta$  and  $x = e^{C/\eta}$ , where  $C$  is a large enough constant, implies that

$$\begin{aligned} \sum_{\substack{n \leq e^{C/\eta} \\ P^-(n) > y}} \frac{(1 * f)(n)}{n^{1-\eta}} &= \left\{ \left( -\frac{1}{\eta} + \gamma_{1-\eta,y} \right) L_y(1 - \eta, f) + \frac{e^C L_y(1, f)}{\eta} \right\} \prod_{p \leq y} \left( 1 - \frac{1}{p} \right) + O\left( \frac{1}{C} \right) \\ &\leq \frac{e^C}{\eta} L_y(1, f) \prod_{p \leq y} \left( 1 - \frac{1}{p} \right) + \frac{1}{2}. \end{aligned}$$

On the other hand, the sum on the left hand side of the above inequality is at least  $(1 * f)(1) = 1$ , by positivity (our assumptions that  $f(p) \in \mathbb{R}$  for  $p > Q$  and that  $f$  is completely multiplicative imply that  $(1 * f)(n) \geq 0$  for  $n$  with  $P^-(n) > Q$ ). So we find that

$$L_y(1, f) \geq \frac{\eta}{2e^C} \prod_{p \leq y} \left( 1 - \frac{1}{p} \right)^{-1} \asymp \eta \log y \asymp 1.$$

If  $Q'$  is as in Theorem 2.4, then the above relation and Theorem 2.4 imply that  $\log y \gg \log Q'$ . Since we also have that  $\log Q' \asymp (\log Q)/L_Q(1, f)$  and  $\log y \asymp 1/\eta$ , (7.3) follows.

Fix now a small enough constant  $c \leq 1/M^2$ . Note that  $L_Q(\sigma, f) \geq 0$  for  $\sigma > 1$ , by the Euler product representation. So if  $L_Q(s, f)$  does not vanish in  $[1 - \sqrt{c}/\log Q, 1]$ , then we must have that  $L(1 - \sqrt{c}/\log Q, f) > 0$  by continuity, and (7.3) gives us that  $L_Q(1, f) \geq c_1 \sqrt{c}$  for some positive constant  $c_1$  that is independent of  $c$ . Consequently, Lemma 4.2(a) implies that for  $\sigma \in \mathbb{R}$  with  $|\sigma - 1| \leq c/\log Q$

$$(7.4) \quad L_Q(\sigma, f) = L_Q(1, f) + \int_1^\sigma L'_Q(u, f) du = L_Q(1, f) + O(|1 - \sigma| \log Q) \geq c_1 \sqrt{c} + O(c) \gg \sqrt{c},$$

provided that  $c$  is small enough. Since we also have that  $L_Q(\sigma, f) \ll 1$  by Lemma 4.2(a), the theorem follows in this case.

Lastly, consider the case that  $L_Q(s, f)$  has a zero in  $[1 - \sqrt{c}/\log Q, 1]$ , say at  $\beta$ . Relation (7.4) with  $\sigma = \beta$  and Lemma 4.2(a) imply that

$$L_Q(1, f) = \int_\beta^1 L'_Q(u, f) du \ll (1 - \beta) \log Q \ll \sqrt{c}.$$

So if we let  $x = Q^{1/c^{1/4}}$ ,  $y = Q$  and  $s = 1$  in Lemma 4.1(b), we obtain the estimate

$$\begin{aligned} 1 &\leq \sum_{\substack{n \leq Q^{1/c^{1/4}} \\ P^-(n) > Q}} \frac{(1 * f)(n)}{n} = \left\{ \left( \frac{\log Q}{c^{1/4}} + \gamma_{1,Q} \right) L_Q(1, f) + L'_Q(1, f) \right\} \prod_{p \leq Q} \left( 1 - \frac{1}{p} \right) + O(c^{1/4}) \\ &= L'_Q(1, f) \prod_{p \leq Q} \left( 1 - \frac{1}{p} \right) + O(c^{1/4}), \end{aligned}$$

since  $\gamma_{1,y} \ll \log y$  by Lemma 3.5. So if  $c$  is small enough, then  $L'_Q(1, f) \geq c_0 \log Q$  for some absolute positive constant  $c_0$ . Consequently, for  $u \in [1 - \sqrt{c}/\log Q, 1 + \sqrt{c}/\log Q]$  Lemma 4.2(a) implies that

$$L'_Q(u, f) = L'_Q(1, f) - \int_{\sigma}^1 L''_Q(w, f) dw \geq c_0 \log Q + O(|1 - \sigma| \log^2 Q) \geq \frac{c_0 \log Q}{2},$$

provided that  $c$  is small enough. Since  $L_Q(\sigma, f) = \int_{\beta}^{\sigma} L'_Q(u, f) du$  for  $\sigma > 1 - 1/\log Q$ , and the zeroes of  $L(s, f)$  and  $L_Q(s, f)$  are in one-to-one correspondence by relation (7.1), the theorem follows.  $\square$

## 8. PROOF OF THEOREMS 1.2 AND 2.1

In this section we show Theorems 1.2 and 2.1. This will be done in various steps which are split among three subsections.

**8.1. Technical preparation.** In this subsection we list some technical results that we will use in the proof of Theorems 1.2 and 2.1. Here and for the rest of the paper, given an arithmetic function  $f : \mathbb{N} \rightarrow \mathbb{C}$ ,  $k \geq 0$ ,  $x \geq 1$  and  $\sigma > 1$ , we set

$$S_k(x; f) = \sum_{n \leq x} f(n) (\log n)^k \quad \text{and} \quad I_k(\sigma; f) = \left( \int_0^{\infty} \left| \frac{S_k(e^u; f)}{e^{\sigma u}} \right|^2 dt \right)^{1/2}.$$

**Lemma 8.1.** *Let  $f : \mathbb{N} \rightarrow \mathbb{C}$  such that  $\sum_{n \leq y} |f(n)| \leq cy$  for all  $y \geq 1$ , for some constant  $c \geq 1$ . For  $x \geq 2$ ,  $r, k \in \mathbb{N} \cup \{0\}$  and  $\sigma \in (1, 1 + 1/\log x]$  we have that*

$$|(\log x)^k S_0(x; f) - S_k(x; f)| \ll \frac{(k+r)2^{k+r} x}{(\log x)^{r+1}} I_{k+r}(\sigma; f) + c \cdot (k+r) \sqrt{x} (\log x)^{k-1}.$$

*Proof.* By partial summation we have that

$$\begin{aligned} (\log x)^k S_0(x; f) - S_k(x; f) &= (\log x)^k \int_{2^-}^x \frac{dS_{k+r}(t; f)}{(\log t)^{k+r}} - \int_{2^-}^x \frac{dS_{k+r}(t; f)}{(\log t)^r} \\ &= \int_2^x \left( \frac{(k+r)(\log x)^k}{(\log t)^{k+r+1}} - \frac{r}{(\log t)^{r+1}} \right) \frac{S_{k+r}(t; f)}{t} dt. \end{aligned}$$

When  $t \leq \sqrt{x}$  we use the trivial bound  $|S_{k+r}(t; f)| \leq ct(\log t)^{k+r}$ . Finally, for  $t \in [\sqrt{x}, x]$  we note that

$$\frac{|S_{k+r}(t; f)|}{t} \ll \frac{|S_{k+r}(t; f)|}{t^{\frac{1+2\sigma}{2}}} \cdot \sqrt{t},$$

since we have assumed that  $\sigma \in (1, 1 + 1/\log x]$ . So the Cauchy-Schwarz inequality and the substitution  $t = e^u$  complete the proof of the lemma.  $\square$

**Lemma 8.2.** *Let  $f : \mathbb{N} \rightarrow \mathbb{C}$  be such that  $\sum_{x-y < n \leq x} |f(n)| \leq cy$  for all  $x \geq 1$  and all  $y \in (\sqrt{x}, x]$ , where  $c$  is some constant. For  $x \geq 2$ ,  $k \in \mathbb{N} \cup \{0\}$  and  $\sigma \in (1, 1 + 1/\log x]$  we have that*

$$\frac{1}{x} \sum_{n \leq x} f(n) \ll c4^k \left\{ \frac{1}{(\log x)^k} \int_{\mathbb{R}} |L^{(k)}(\sigma + it, f)| \frac{dt}{1+t^2} \right\}^{1/2} + \frac{c4^k}{\sqrt{x}}.$$

*Proof.* For every  $y \geq 1$  we have that

$$T(y) := \sum_{n \leq y} f(n)(\log n)^k \log \frac{y}{n} = \frac{1}{2\pi i} \int_{\Re(s)=\sigma} L^{(k)}(s, f) \frac{y^s}{s^2} d\tau \ll y^\sigma \int_{\mathbb{R}} \frac{|L^{(k)}(\sigma + it, f)|}{t^2 + 1} dt.$$

Set

$$E(y) = \max_{y \leq u \leq 2y} \left\{ \frac{|T(u)|}{cu(\log u)^k} \right\} \leq 1 \quad \text{and} \quad \Delta(y) = \max\{y\sqrt{E(y)}, \sqrt{y}\}.$$

Then we find that

$$\begin{aligned} S_k(y; f) &= \frac{1}{\log \frac{y+\Delta(y)}{y}} \sum_{n \leq y} f(n)(\log n)^k \log \frac{y+\Delta(y)}{y} \\ &= \frac{1}{\log \frac{y+\Delta(y)}{y}} (T(y+\Delta(y)) - T(y)) + O(c\Delta(y) \log^k(2y)) \\ &\ll c \left( \frac{y^2 E(y)}{\Delta(y)} + \frac{\Delta(y)}{y} \right) (2 \log y)^k \ll c\Delta(y) (2 \log y)^k \\ &\ll c \cdot y (2 \log y)^k \left\{ \frac{1}{(\log y)^k} \int_{\mathbb{R}} |L^{(k)}(\sigma + it, f)| \frac{dt}{1+t^2} \right\}^{1/2} + c\sqrt{y} (2 \log y)^k \end{aligned}$$

for all  $y \leq x$ , since  $\sigma \leq 1 + 1/\log x$ . The lemma then follows by the relation

$$S_0(x; f) = O(c\sqrt{x}) + \int_{\sqrt{x}}^x \frac{dS_k(u; f)}{(\log u)^k}$$

and integration by parts. □

Here and for the rest of the paper given  $y \geq 0$  we define

$$\alpha(y) = \begin{cases} 1 & \text{if } y < 1/2, \\ 0 & \text{if } y \geq 1/2. \end{cases}$$

**Lemma 8.3.** *Let  $f : \mathbb{N} \rightarrow \mathbb{U}$  be a completely multiplicative function,  $Q \geq 3$ ,  $1 \leq \tau \leq \exp \left\{ \frac{(\log Q)^{3/2}}{2000\sqrt{\log \log Q}} \right\}$  and  $A \geq 3$  such that*

$$\left| \sum_{n \leq x} f(n)n^{it} \right| \leq \frac{x(\log Q)^{A-2}}{(\log x)^A} + \frac{x^{1-1/\log Q}}{(\log x)^2} \quad (x \geq Q, t \in [-\tau, \tau]).$$

*Let  $\sigma > 1$ ,  $J \subset [-\tau, \tau]$ ,  $\log y = \max\{\log Q, (\log Q)/\min_{t \in J} |L_Q(\sigma + it, f)|\}$ , and  $k \in \mathbb{N} \cap [1, A-2]$ .*

(a) If  $\sigma \leq 1 + 1/\log y$ , then for  $F(s) = \frac{1}{L}(s, f)$  and for  $F(s) = \frac{L'}{L}(s, f)$  we have that

$$\begin{aligned} \int_J |F^{(k)}(\sigma + it)| dt &\ll (ck \log y)^k \left( \log \frac{2}{(\sigma - 1) \log y} \right)^{\alpha(A-k-2)} \\ &\quad + \tau(ck \log Q)^{k+1} \left( \log \frac{2}{(\sigma - 1) \log Q} \right)^{\alpha(A-k-2)}. \end{aligned}$$

(b) If  $k \leq A - 3$ ,  $\sigma \leq 1 + 1/\log Q$  and  $\liminf_{u \rightarrow \infty} \sum_{p \leq u} \Re(f(p))/p = -\infty$ , then

$$\left( \frac{L'}{L} \right)^{(k)} (\sigma + it, 1 * f) \ll (ck \log Q)^{k+1} \left( \log \frac{2}{(\sigma - 1) \log Q} \right)^{\alpha(A-k-3)}.$$

*Proof.* (a) For  $z \in [Q, y]$ ,  $1 \leq j \leq m \leq k + 1$  and  $t \in J$  Lemma 4.2(a), applied with  $f(n)n^{-it}$  in place of  $f$ , implies that

$$\begin{aligned} |L_z^{(j)}(\sigma + it)| &\leq j!(c_1 \log z)^j \left( \log \frac{2}{(\sigma - 1) \log z} \right)^{\alpha(A-j-1)} \\ &\leq j! \left\{ c_1(\log z) \left( \log \frac{2}{(\sigma - 1) \log z} \right)^{\frac{\alpha(A-m-1)}{m}} \right\}^j \end{aligned}$$

for some constant  $c_1 > 0$ . So Lemma 5.1 gives us that

$$(8.1) \quad \left| \left( \frac{L'}{L} \right)^{(m-1)} (\sigma + it) \right| \ll m! \left\{ \frac{c_2(\log z) \left( \log \frac{2}{(\sigma-1) \log z} \right)^{\frac{\alpha(A-m-1)}{m}}}{|L_z(\sigma + it, f)|} \right\}^m \quad (1 \leq m \leq k + 1)$$

for some  $c_2 > 0$ . Next, for all  $t \in J$ , if we set  $\log y_t = \max\{\log Q, (\log Q)/|L_Q(\sigma + it, f)|\}$ , then we have that  $|L_{y_t}(\sigma + it, f)| \asymp 1$ , by Theorem 2.2. So applying (8.1) with  $z = y_t \in [Q, y]$  yields that

$$(8.2) \quad \left| \left( \frac{L'}{L} \right)^{(m-1)} (\sigma + it) \right| \leq m! \left\{ c_3(\log y_t) \left( \log \frac{2}{(\sigma - 1) \log y_t} \right)^{\frac{\alpha(A-m-1)}{m}} \right\}^m \quad (1 \leq m \leq k + 1).$$

Since  $\alpha(A - m - 1) = 0$  for  $m \leq k \leq A - 2$ , (8.2) and Lemma 3.1 with  $F(s) = 1/L(s, f)$  imply that

$$(8.3) \quad \left| \frac{(1/L)^{(k)}}{1/L} (\sigma + it) \right| \leq k!(2c_3 \log y_t)^k.$$

Furthermore, note that

$$|L(\sigma + it, f)| = \left| L_{y_t}(\sigma + it, f) \sum_{P^+(n) \leq y_t} \frac{f(n)}{n^{\sigma+it}} \right| \asymp \left| \sum_{P^+(n) \leq y_t} \frac{f(n)}{n^{\sigma+it}} \right| \gg \frac{1}{\log y_t}.$$

So we find that

$$(8.4) \quad \left| \left( \frac{1}{L} \right)^{(k)} (\sigma + it, f) \right| \ll k!(2c_3 \log y_t)^{k+1}.$$

In any case, relations (8.2) and (8.4) imply that

$$|F^{(k)}(\sigma + it)| \leq k!(c_4 \log y_t)^{k+1} \left( \log \frac{2}{(\sigma - 1) \log y_t} \right)^{\alpha(A-k-2)}.$$

Finally, if  $t_0 \in [-\tau, \tau]$  is such that  $\log y_{t_0} = \log y = \max_{t \in J} \log y_t$ , then Theorem 2.3 implies that

$$(8.5) \quad \log y_t \asymp \begin{cases} \log y & \text{if } |t - t_0| \leq 1/\log y, \\ |t - t_0|^{-1} & \text{if } 1/\log y \leq |t - t_0| \leq 1/\log Q, \\ \log Q & \text{if } |t - t_0| \geq 1/\log Q, \end{cases}$$

and part (a) of the lemma follows.

(b) Assume that  $1 \leq k \leq A - 3$  and  $\liminf_{u \rightarrow \infty} \sum_{p \leq u} \Re(f(p))/p = -\infty$ . Fix  $t \in [-\tau, \tau]$ . We claim that

$$(8.6) \quad |L_Q(\sigma + it, f)| \asymp \begin{cases} (\sigma - 1) \log Q & \text{if } |t| \leq \sigma - 1, \\ |t| \log Q & \text{if } \sigma - 1 \leq |t| \leq 1/\log Q, \\ 1 & \text{if } |t| \geq 1/\log Q. \end{cases}$$

Indeed, Lemma 4.2(b) implies that  $|L_Q(\sigma, 1 * f)| \ll 1$  and, consequently,

$$|L_Q(\sigma, f)| \ll \frac{1}{\zeta_Q(\sigma)} \asymp (\sigma - 1) \log Q.$$

Since for every  $t_1 \in \mathbb{R}$  we have that

$$|L_Q(\sigma + it_1, f)| \geq \frac{1}{\zeta_Q(\sigma)} \asymp (\sigma - 1) \log Q$$

trivially, we deduce that

$$(\sigma - 1) \log Q \asymp |L_Q(\sigma, f)| \asymp \min \{|L_Q(\sigma + it_1)| : t_1 \in [-\tau, \tau]\}.$$

So (8.6) follows immediately by Theorem 2.3.

Next, observe that relations (5.7) and (5.8) imply (8.6) with  $\mu$  in place of  $f$ , since  $Q \geq V_\tau^{100} \geq V_t^{100}$  by our assumption on the size of  $\tau$ . So we deduce that

$$(8.7) \quad |L_Q(\sigma + it, 1 * f)| = \left| \frac{L_Q(\sigma + it, f)}{L_Q(\sigma + it, \mu)} \right| \asymp 1 \quad (t \in [-\tau, \tau]).$$

Moreover, we claim that

$$(8.8) \quad \begin{aligned} |L_Q^{(j)}(\sigma + it, 1 * f)| &\leq (c_3 j \log Q)^j \left( \log \frac{2}{(\sigma - 1) \log Q} \right)^{\alpha(A-j-2)} \\ &\leq \left\{ c_3 j (\log Q) \left( \log \frac{2}{(\sigma - 1) \log Q} \right)^{\frac{\alpha(A-k-3)}{k+1}} \right\}^j \quad (0 \leq j \leq k + 1). \end{aligned}$$

Indeed, if  $|t| \leq 1$ , then (8.8) follows by Lemma 4.2(b). On the other hand, if  $|t| > 1$ , then we note that

$$L_Q^{(j)}(\sigma + it, 1 * f) = \sum_{r=0}^j \binom{r}{j} L_Q^{(j-r)}(\sigma + it, f) \zeta_Q^{(r)}(\sigma + it).$$

We bound  $L_Q^{(j-r)}(\sigma + it, f)$  by Lemma 4.2(a) and  $\zeta_Q^{(j-r)}(\sigma + it)$  by the first formula in Lemma 3.5 and partial summation, which is possible by our assumption that  $|t| \leq \tau \leq \exp \left\{ \frac{(\log Q)^{3/2}}{2000\sqrt{\log \log Q}} \right\}$ . So we find that (8.8) holds when  $|t| > 1$  too.

Relations (8.7), (8.8) and Lemma 5.1 imply that

$$\left( \frac{L'}{L} \right)^{(k)} (\sigma + it, 1 * f) \ll (c_4 j \log y)^{k+1} \left( \log \frac{2}{(\sigma - 1) \log Q} \right)^{\alpha(A-k-3)},$$

which completes the proof of the lemma.  $\square$

**Lemma 8.4.** *Let  $f, Q, \tau$  and  $A$  be as in Lemma 8.3. Consider  $\sigma > 1$ ,  $k \in \mathbb{Z} \cap [0, A-2]$ ,  $J \subset [-\tau, \tau]$  and  $q \in [1, Q]$ .*

(a) *We have that*

$$(8.9) \quad \frac{1}{k!^2} \int_J \left| \left( \frac{1}{L} \right)^{(k)} (\sigma + it, f) \right|^2 dt \ll \left( \frac{c \log Q}{\min_{t \in J} |L_Q(\sigma + it, f)|} \right)^{2k+1} + \tau (c \log Q)^{2k+1}.$$

*If, in addition,  $k \geq 1$ , then*

$$(8.10) \quad \frac{1}{k!^2} \int_J \left| \left( \frac{L'}{L} \right)^{(k-1)} (\sigma + it, f) \right|^2 dt \ll \left( \frac{c \log Q}{\min_{t \in J} |L_Q(\sigma + it, f)|} \right)^{2k-1} + \tau (c \log Q)^{2k-1} \\ + c^k \sum_{j=1}^k (\log Q)^{2(k-j)} I_j(f),$$

*where  $I_j(f) = I_j(f; \sigma, q, Q, \tau, J)$  is given by*

$$I_j(f) = \min \left\{ \tau (\log Q)^{2j}, \frac{1}{(\sigma - 1)^{2j-1}} + \tau (\log V_\tau)^{2j}, \left( \frac{\log Q}{\log q} \right)^2 \frac{1}{j!^2} \int_J |L_q^{(j)}(\sigma + it, f)|^2 dt \right\}.$$

(b) *If  $\sigma \leq 1 + 1/\log Q$ ,  $k \geq 1$  and  $\liminf_{u \rightarrow \infty} \sum_{p \leq u} \Re(f(p))/p = -\infty$ , then*

$$\int_J \left| \left( \frac{L'}{L} \right)^{(k-1)} (\sigma + it, 1 * f) \right|^2 dt \ll \left\{ \tau (ck \log Q)^{2k-1} + (ck)^{2k} \sum_{j=1}^k (\log Q)^{2(k-j)} I_j(1 * f) \right\} \\ \times \left( \log \frac{2}{(\sigma - 1) \log Q} \right)^{2\alpha(A-k-2)}.$$

*Proof.* (a) Let  $t_0 \in J$  be such that  $|L_Q(\sigma + it_0, f)| = \min_{t \in J} |L_Q(\sigma + it, f)|$ . Set  $\mathcal{A} = J \cap [t_0 - 1/\log Q, t_0 + 1/\log Q]$  and  $\mathcal{B} = J \setminus \mathcal{A}$ . Relations (8.2), (8.4) and (8.5) imply that

$$\int_{\mathcal{A}} \left| \left( \frac{1}{L} \right)^{(k)} (\sigma + it, f) \right|^2 dt \ll \left( \frac{c_1 k \log Q}{\min_{t \in J} |L_Q(\sigma + it, f)|} \right)^{2k+1}$$

and

$$\int_{\mathcal{A}} \left| \left( \frac{L'}{L} \right)^{(k-1)} (\sigma + it, f) \right|^2 dt \ll \left( \frac{c_1 k \log Q}{\min_{t \in J} |L_Q(\sigma + it, f)|} \right)^{2k-1}.$$

So it remains to bound the contributions to the integrals from  $t \in \mathcal{B}$ .

First, we handle the integral involving  $1/L(s, f)$ . Relations (8.3) and (8.5) imply that, for any  $t \in \mathcal{B}$ , we have that

$$(8.11) \quad |L_Q(\sigma + it, f)| \asymp 1 \quad \text{and} \quad \frac{(1/L)^{(k)}}{1/L}(\sigma + it, f) \ll (c_2 k \log Q)^k.$$

So

$$\left(\frac{1}{L}\right)^{(k)}(\sigma + it, f) \ll (c_3 k \log Q)^k \left| \sum_{P^+(n) \leq Q} \frac{\mu(n)f(n)}{n^{\sigma+it}} \right|.$$

Since

$$\log \left| \sum_{P^+(n) \leq Q} \frac{\mu(n)f(n)}{n^{\sigma+it}} \right| = - \sum_{p \leq Q} \frac{\Re(f(p)p^{-it})}{p^\sigma} + O(1) = \log \left| L \left( \sigma + \frac{1}{\log Q} + it, \mu f \right) \right| + O(1),$$

by Lemma 6.1, we find that

$$\int_{\mathcal{B}} \left| \left(\frac{1}{L}\right)^{(k)}(\sigma + it, f) \right|^2 dt \ll (c_4 k \log Q)^{2k} \int_{-\tau}^{\tau} \left| L \left( \sigma + \frac{1}{\log Q} + it, \mu f \right) \right|^2 dt.$$

We break the range of integration into intervals of length at most 1 and we observe that

$$\begin{aligned} \int_z^{z+1} \left| L \left( \sigma + \frac{1}{\log Q} + it, \mu f \right) \right|^2 dt &\leq 3 \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \zeta \left( 1 + \frac{1}{\log Q} + it \right) \right|^2 dt \ll \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dt}{(|t| + 1/\log Q)^2} \\ &\ll \log Q \end{aligned}$$

for every  $z \in \mathbb{R}$ , by Lemmas 3.2 and 3.5, which proves (8.9), since  $J$  is arbitrary.

It remains to show (8.10). Note that

$$(8.12) \quad \begin{aligned} \int_{\mathcal{B}} \left| \left(\frac{L'}{L}\right)^{(k-1)}(\sigma + it, f) - \left(\frac{L'_q}{L_q}\right)^{(k-1)}(\sigma + it, f) \right|^2 dt &\ll \tau \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \sum_{P^+(n) \leq q} \frac{\Lambda(n)(\log n)^{k-1}}{n^{\sigma+it}} \right|^2 dt \\ &\ll \tau (c_5 k \log q)^{2k-1}, \end{aligned}$$

by covering  $\mathcal{B}$  by  $O(\tau)$  intervals of length 1 and applying Lemmas 3.2 and 3.6 to each one of them. Next, Lemma 3.1 implies that

$$(8.13) \quad \int_{\mathcal{B}} \left| \left(\frac{L'_q}{L_q}\right)^{(k-1)}(\sigma + it, f) \right|^2 dt \leq \sum_{j=1}^k \frac{64^k k!^2}{j!^{\frac{2k}{j}}} \int_{\mathcal{B}} \left| \frac{L_q^{(j)}}{L_q}(\sigma + it, f) \right|^{\frac{2k}{j}} dt.$$

Before we proceed, we need an upper bound for  $(L_q^{(j)}/L_q)(\sigma + it, f)$ . Observe that

$$\left| \frac{L_q^{(m)}}{L_q}(\sigma + it, f) \right| \leq (c_6 m \log Q)^m \quad (t \in B, 1 \leq m \leq k)$$

by the first inequality in relation (8.11) and Lemma 4.2(a). Moreover, if we set  $F(s) = \sum_{p|n \Rightarrow q < p \leq Q} f(n)/n^s$ , then

$$\frac{F^{(m)}}{F}(\sigma + it, f) \ll (c_7 m \log Q)^m \quad (t \in B, 1 \leq m \leq k)$$

by Lemma 3.6 and our assumption that  $f$  is completely multiplicative. Since  $L_q(s, f) = L_Q(s, f)F(s)$ , we deduce that

$$(8.14) \quad \frac{L_q^{(j)}(\sigma + it, f)}{L_q} = \sum_{m=0}^j \binom{j}{m} \frac{L_Q^{(j-m)}(\sigma + it, f)}{L_Q} \frac{F^{(m)}(\sigma + it)}{F} \ll (c_8 j \log Q)^j \quad (t \in B, 1 \leq j \leq k).$$

Consequently,

$$(8.15) \quad \int_B \left| \frac{L_q^{(j)}(\sigma + it, f)}{L_q} \right|^{\frac{2k}{j}} dt \leq (c_9 j \log Q)^{2(k-j)} \int_B \left| \frac{L_q^{(j)}(\sigma + it, f)}{L_q} \right|^2 dt.$$

It remains to bound the integral on the right hand side of (8.15), which we perform in three different ways. First, we have the trivial bound

$$(8.16) \quad \int_B \left| \frac{L_q^{(j)}(\sigma + it, f)}{L_q} \right|^2 dt \ll \tau(c_6 j \log Q)^{2j},$$

which follows immediately by (8.14). Next, Lemmas 3.2 and 3.6 imply the bound

$$(8.17) \quad \int_B \left| \frac{L_q^{(j)}(\sigma + it, f)}{L_q} \right|^2 dt \leq 3 \int_{-\tau}^{\tau} \left| \frac{\zeta^{(j)}(\sigma + it)}{\zeta} \right|^2 dt \ll \tau(c_9 j \log V_\tau)^{2j} + \left( \frac{c_9 j}{\sigma - 1} \right)^{2j-1}.$$

Finally, observe that

$$|L_q(\sigma + it)| \gg \frac{\log q}{\log Q} |L_Q(\sigma + it)| \gg \frac{\log q}{\log Q} \quad (t \in \mathcal{B} \subset J)$$

by (8.11). So we deduce that

$$(8.18) \quad \int_B \left| \frac{L_q^{(j)}(\sigma + it, f)}{L_q} \right|^2 dt \ll \left( \frac{\log Q}{\log q} \right)^2 \int_J |L_q^{(j)}(\sigma + it, f)|^2 dt.$$

Combining relations (8.16), (8.17) and (8.18), we find that

$$\int_B \left| \frac{L_q^{(j)}(\sigma + it, f)}{L_q} \right|^2 dt \ll (c_{10} j)^j I_j(f).$$

Combining the above estimate with (8.12), (8.13) and (8.15) completes the proof of (8.10) and hence of part (a).

(b) The claimed estimate follows by relations (8.7) and (8.8) and the argument above leading to (8.10). Note here that in the course of the argument, we will need an upper bound on  $(G^{(m)}/G)(\sigma + it)$ , where  $G(s) = \sum_{p|n \Rightarrow q < p \leq Q} (1 * f)(n)/n^s = F(s) \sum_{p|n \Rightarrow q < p \leq Q} 1/n^s$ . This can be done by combining Lemma 3.6 with the identity

$$\frac{G^{(m)}(s)}{G}(s) = (-1)^m \sum_{r=0}^m \binom{r}{m} \sum_{p|n \Rightarrow q < p \leq Q} \frac{\Lambda_r(n) f(n)}{n^s} \sum_{p|n \Rightarrow q < p \leq Q} \frac{\Lambda_{m-r}(n)}{n^s}.$$

□



**8.2. Two intermediate results.** In the proof of Theorems 1.2 and 2.1 a crucial role is played by the following two estimates.

**Theorem 8.5.** *Let  $f : \mathbb{N} \rightarrow \mathbb{U}$  be a completely multiplicative function satisfying (2.1) for some  $Q \geq 3$ ,  $A \geq 3$  and  $\delta > 0$ . Consider  $k \in \mathbb{N} \cap [2, A-1]$ ,  $x \geq 2$  and  $T \geq 1$  and define  $y$  and  $Y$  by  $\log Y = (\log x)/(\log Q + k)$  and  $\log y = e^{N_f(x;T)}/(\log Q + k)$ , where  $N_f(x;T)$  is given by (2.4).*

(a) *For  $g \in \{\mu f, \Lambda f\}$  there is some constant  $c = c(\delta)$  such that*

$$\frac{S_0(x; g)}{x} \ll (ck)^{k/2} \left( \frac{(\log \log y)^{\alpha(A-k-1)}}{(\log y)^{k-1}} + \frac{\log x}{T} + \frac{(\log x)(1 + T^{\frac{k}{A-2}-1})(\log \log Y)^{\alpha(|A-k-2|)+\alpha(A-k-1)}}{(\log Y)^k} \right)^{1/2}.$$

(b) *If  $k \leq A-2$  and  $\liminf_{u \rightarrow \infty} \sum_{p \leq u} \Re(f(p))/p = -\infty$ , then*

$$\frac{S_0(x; \Lambda(1 * f))}{x} \ll \frac{(ck)^{k/2} \sqrt{\log x} (\log \log Y)^{\alpha(A-k-2)}}{(\log Y)^{k/2}}.$$

*Proof.* (a) We impose the condition  $1 \leq T \leq (\log Y)^k$ , since the result is trivial when  $\log Y < 1$  and the case  $T > (\log Y)^k$  follows by the case  $T = (\log Y)^k$ . Also, we may assume that  $\delta \leq 1/2$ .

First, note that summation by parts implies that

$$\begin{aligned} \sum_{n \leq w} f(n) n^{-it} &= O(\sqrt{w}) + \int_{\sqrt{w}}^w u^{-it} dS_0(u, f) \\ &\ll \frac{2^A(|t|+1)(\log Q)^{A-2}w}{(\log w)^A} + (|t|+1)w^{1-\delta} \quad (w \geq Q^2). \end{aligned}$$

Recall the definition of  $Q_t$  by (2.2). If  $w \geq Q_t$ , then

$$(|t|+1)w^{1-\delta} \leq w^{1-3\delta/4} \ll_{\delta} \frac{w^{1-\delta/2}}{(\log w)^2} \leq \frac{w^{1-1/\log Q_t}}{(\log w)^2}.$$

Since  $Q_t \geq Q^2$ , we find that

$$(8.19) \quad \sum_{n \leq w} f(n) n^{-it} \ll_{\delta} \frac{w(\log Q_t)^{A-2}}{(\log w)^A} + \frac{w^{1-1/\log Q_t}}{(\log w)^2} \quad (w \geq Q_t).$$

Set  $\sigma = 1 + 1/\log x$  and consider  $g \in \{\mu f, \Lambda f\}$ . Lemma 8.2 implies that

$$(8.20) \quad \frac{S_0(g; x)}{x} \ll 4^k \left\{ \frac{1}{(\log x)^{k-1}} \int_{\mathbb{R}} |L^{(k-1)}(\sigma + it, g)| \frac{dt}{1+t^2} \right\}^{1/2} + \frac{4^k}{\sqrt{x}}.$$

For  $\tau \geq 1$  set  $J_\tau = \{t \in \mathbb{R} : \tau/2 - 1 \leq |t| \leq \tau - 1\}$ . Note that

$$\begin{aligned}
 (8.21) \quad N_f(x; T) &= O(1) + \min_{t \in [-T, T]} \left\{ \log \log(\min\{Q_t, x\}) + \sum_{Q_t < p \leq x} \frac{1 + \Re(f(p))}{p} \right\} \\
 &= O(1) + \log \log x + \min_{1 \leq \tau \leq T+1} \min_{t \in J_\tau} \sum_{Q_\tau < p \leq x} \frac{\Re(f(p))}{p} \\
 &= O(1) + \log \log x + \min_{1 \leq \tau \leq T+1} \min_{t \in J_\tau} \log |L_{Q_\tau}(\sigma + it, f)|,
 \end{aligned}$$

by Lemma 6.1 and the fact that, for any  $t \in J_\tau$ ,  $Q_t = Q_\tau^{O(1)}$ . So when  $\tau \in [1, T+1]$ , Lemma 8.3(a), with  $Q_\tau$  in place of  $Q$ ,  $J_\tau$  in place of  $J$  and  $k-1$  in place of  $k$ , implies that

$$\begin{aligned}
 (8.22) \quad \int_{J_\tau} |L^{(k-1)}(\sigma + it, g)| dt &\ll \left( \frac{c_1 k (\log x) (\log Q_\tau)}{e^{N_f(x; T)}} \right)^{k-1} \log^{\alpha(A-k-1)} \left( 2 + \frac{e^{N_f(x; T)}}{\log Q_\tau} \right) \\
 &\quad + \tau (c_1 k \log Q_\tau)^k \log^{\alpha(A-k-1)} \left( 2 + \frac{\log x}{\log Q_\tau} \right)
 \end{aligned}$$

for some positive constant  $c_1 = c_1(\delta)$ . Moreover, we have the trivial bound  $L^{(k-1)}(\sigma + it, g) \ll (c_2 k \log x)^k$ , which implies that

$$(8.23) \quad \int_{J_\tau} |L^{(k-1)}(\sigma + it, f)| dt \ll \tau (c_2 k \log x)^k.$$

Finally, we break the range of integration in (8.20) as  $\mathbb{R} = \bigcup_{m \geq 0} J_{2^m}$ . If  $2^m \leq T$ , we bound the integral over  $J_{2^m}$  by (8.22); else, we use (8.23). Summing these estimates over  $m \geq 0$  completes the proof of part (a) of the theorem.

(b) The claimed estimate follows by the argument leading to part (a) on using Lemma 8.3(b) in place of Lemma 8.3(a).  $\square$

**Theorem 8.6.** *Consider a completely multiplicative function  $f : \mathbb{N} \rightarrow \mathbb{U}$  satisfying (2.1) for some  $Q \geq 3$ ,  $A \geq 3$  and  $\delta > 0$ . Let  $k \in \mathbb{N} \cap [0, (A-3)/2]$ ,  $x \geq 2$ ,  $T \geq 1$  and  $\sigma = 1 + 1/\log x$  and define  $y$  and  $Y$  by  $\log y = e^{N_f(x; T)}/(\log Q + k)$  and  $\log Y = (\log x)/(\log Q + k)$ , where  $N_f(x; T)$  is given by (2.4).*

(a) *There is some constant  $c = c(\delta)$  such that*

$$(8.24) \quad \frac{I_k(\sigma; \mu f)}{k! (\log x)^{k+\frac{1}{2}}} \ll c^k \frac{(\log \log(y+3))^{\frac{\alpha(A-2k-3)}{2}}}{(\log y)^{k+\frac{1}{2}}} + \frac{c^k}{T}$$

and

$$(8.25) \quad \frac{I_k(\sigma; \Lambda f)}{k! (\log x)^{k+\frac{1}{2}}} \ll \frac{c^k}{(\log y)^{k+\frac{1}{2} - \frac{(k+1)(k+2)}{4(A-1)}}} + \frac{c^k}{T}.$$

(b) *If  $\liminf_{u \rightarrow \infty} \sum_{p \leq u} \Re(f(p))/p = -\infty$ , then*

$$\frac{I_k(\sigma; \Lambda(1 * f))}{k! (\log x)^{k+\frac{1}{2}}} \ll c^k \frac{(\log \log(Y+3))^{\alpha(A-k-3)}}{(\log Y)^{k+\frac{1}{2} - \frac{(k+1)(k+2)}{4(A-1)}}} + \frac{c^k}{T}.$$

*Proof.* (a) We impose the condition  $1 \leq T \leq (\log y)^{k+\frac{1}{2}}$ , since the result is trivial when  $\log y < 1$  and the case  $T > (\log y)^{k+\frac{1}{2}}$  follows by the case  $T = (\log y)^{k+\frac{1}{2}}$ . First, we show (8.24). Note that the Fourier transform of the function  $u \rightarrow e^{-\sigma u} S_k(e^u; \mu f)$  is the function  $\xi \rightarrow \frac{(-1)^k (1/L)^{(k)}(\sigma+2\pi i\xi, f)}{\sigma+2\pi i\xi}$ . So Plancherel's identity implies that

$$(8.26) \quad \int_0^\infty \left| \frac{S_k(e^u; \mu f)}{e^{\sigma u}} \right|^2 du = \frac{1}{2\pi} \int_{\mathbb{R}} \left| \left( \frac{1}{L} \right)^{(k)} (\sigma + it, f) \right|^2 \frac{dt}{\sigma^2 + t^2}.$$

For  $\tau \geq 1$  set  $J_\tau = \{t \in \mathbb{R} : \tau/2 - 1 \leq |t| \leq \tau - 1\}$ . We break the range of integration in the right hand side of (8.26) into sets of the form  $J_{2^m}$  for  $m \in \mathbb{N}$  and bound the contribution of each one of them individually. First, note that for every  $\tau \geq 1$  we have that

$$(8.27) \quad \int_{J_\tau} \left| \left( \frac{1}{L} \right)^{(k)} (\sigma + it, f) \right|^2 dt \leq \int_{-\tau}^{\tau} |\zeta^{(k)}(\sigma + it)|^2 dt \ll c_3^k k!^2 \int_{-\tau}^{\tau} \left( \frac{1}{\sigma - 1 + |t|} + \log V_t \right)^{2k+2} dt \\ \ll k!^2 ((c_4 \log x)^{2k+1} + \tau (c_4 \log V_\tau)^{2k+2}),$$

by Lemmas 3.2 and 3.5. Moreover, if  $\tau \in [1, T]$ , then Lemma 8.4, with  $J_\tau$  in place of  $J$  and  $Q_\tau$  in place of  $Q$ , and relation (8.21) imply that

$$(8.28) \quad \frac{1}{k!^2} \int_{J_\tau} \left| \left( \frac{1}{L} \right)^{(k)} (\sigma + it, f) \right|^2 dt \ll (c_5 k \log Q_\tau)^{2k+1} \left( \left( \frac{\log x}{e^{N_f(x; T)}} \right)^{2k+1} + \tau \right)$$

for some positive constant  $c_5 = c_5(\delta)$ . We partition the range of integration in (8.26) as  $\mathbb{R} = \bigcup_{m \in \mathbb{N}} J_{2^m}$  and we apply (8.27) or (8.28) to the part of the integral over  $J_{2^m}$  according to whether  $2^m > T$  or  $2^m \leq T$ , respectively, to find that

$$(8.29) \quad \frac{1}{k!^2} \int_0^\infty \left| \frac{S_k(e^u; \mu f)}{e^{\sigma u}} \right|^2 du \ll \left( \frac{c_6 \log x}{\log y} \right)^{2k+1} + (c_6 \log Q)^{2k+1} \left( \log \frac{\log x}{\log Q} \right)^{\alpha(A-2k-3)} \\ + \frac{(c_6 \log x)^{2k+1}}{T^2}$$

for some  $c_6 = c_6(\delta) > 0$ , which immediately implies (8.24).

Next, we demonstrate (8.25). As before, we have that

$$(8.30) \quad \int_0^\infty \left| \frac{S_k(e^u; \Lambda f)}{e^{\sigma u}} \right|^2 du = \frac{1}{2\pi} \int_{\mathbb{R}} \left| \left( \frac{L'}{L} \right)^{(k)} (\sigma + it, f) \right|^2 \frac{dt}{\sigma^2 + t^2}.$$

We bound the portion over  $t \in J_\tau$  of the above integral in two different ways. First, note that

$$(8.31) \quad \frac{1}{k!^2} \int_{J_\tau} \left| \left( \frac{L'}{L} \right)^{(k)} (\sigma + it, f) \right|^2 dt \ll (c_7 \log x)^{2k+1} + \tau (c_7 \log V_\tau)^{2k+2}$$

by Lemmas 3.6 and 3.2. Moreover, if  $\tau \in [1, T]$ , then relation (8.21) and Lemma 8.4, with  $J_\tau$ ,  $Q_\tau$  and  $Q_0$  in place of  $J$ ,  $Q$  and  $q$ , respectively, implies that

$$\begin{aligned} \frac{1}{k!^2} \int_{J_\tau} \left| \left( \frac{L'}{L} \right)^{(k)} (\sigma + it, f) \right|^2 dt &\ll (c_8 \log Q_\tau)^{2k+1} \left( \left( \frac{\log x}{e^{N(x;T)}} \right)^{2k+1} + \tau \right) \\ &+ c_8^k \sum_{j=1}^{k+1} (\log Q_\tau)^{2(k-j+1)} I_{j,\tau}, \end{aligned}$$

where

$$I_{j,\tau} = \min \left\{ (\log Q_\tau)^{2j}, (\log x)^{2j-1} + \tau (\log V_\tau)^{2j}, \left( \frac{\log Q_\tau}{\log Q_0} \right)^2 \frac{1}{j!^2} \int_{J_\tau} |L_{Q_0}^{(j)}(\sigma + it, f)|^2 dt \right\},$$

for some  $c_8 = c_8(\delta) > 0$ . We separate two cases: if  $\log Q_\tau = 4\delta^{-1} \log(2 + \tau)$ , we use the bound  $I_{j,\tau} \leq (\log Q_\tau)^{2j} = (4\delta^{-1} \log(2 + \tau))^{2j}$ ; else, if  $\log Q_\tau = 2(1 + \tau)^{\frac{1}{A-2}} \log Q$ , we use the bound  $I_{j,\tau} \leq I'_{j,\tau} + \tau (\log V_\tau)^{2j}$ , where

$$I'_{j,\tau} = \min \left\{ \left( \frac{\log x}{\log Q_0} \right)^{2j-1}, 2\tau^{\frac{2}{A-2}} \int_{J_\tau} \frac{|L_{Q_0}^{(j)}(\sigma + it, f)|^2}{j!^2 (\log Q_0)^{2j-1}} dt \right\}.$$

So we deduce that

(8.32)

$$\begin{aligned} \frac{1}{k!^2} \int_{J_\tau} \left| \left( \frac{L'}{L} \right)^{(k)} (\sigma + it, f) \right|^2 dt &\ll (c_9 \log Q_\tau)^{2k+1} \left( \left( \frac{\log x}{e^{N(x;T)}} \right)^{2k+1} + \tau + \tau \frac{(\log V_\tau)^2}{\log Q_\tau} \right) \\ &+ \tau (c_9 \log \tau)^{2k+2} + (c_8 \log Q_0)^{2k+1} \sum_{j=1}^{k+1} \tau^{\frac{2(k-j+1)}{A-2}} I'_{j,\tau} \end{aligned}$$

for some  $c_9 = c_9(\delta) > 0$ . By partitioning the range of integration in (8.26) as  $\mathbb{R} = \bigcup_{m \in \mathbb{N}} J_{2^m}$  and applying (8.27) or (8.28) to the part of the integral over  $J_{2^m}$  according to whether  $2^m > T$  or  $2^m \leq T$ , respectively, we find that

$$\begin{aligned} \frac{1}{k!^2} \int_0^\infty \left| \frac{S_k(e^u; \Lambda f)}{e^{\sigma u}} \right|^2 du &\ll \left( \frac{c_{10} \log x}{\log y} \right)^{2k+1} + (c_{10} \log Q)^{2k+1} \left( \log \frac{\log x}{\log Q} \right)^{\alpha(A-2k-3)} \\ &+ \frac{(c_{10} \log x)^{2k+1}}{T^2} + (c_{10} \log Q_0)^{2k+1} \sum_{j=1}^{k+1} \sum_{\substack{m \in \mathbb{N} \\ 2^m \leq T}} \frac{I'_{j,2^m}}{4^{m(1-\frac{k-j+1}{A-2})}} \end{aligned} \quad (8.33)$$

for some  $c_{10} = c_{10}(\delta) > 0$ . Fix  $j \in \{1, \dots, k+1\}$  and set  $L_j = (\log x / \log Q)^{\frac{(A-2)(2j-1)}{A-1}}$ . Note that

$$\frac{I'_{j,2^m}}{4^{m(1-\frac{k-j+1}{A-2})}} \leq \begin{cases} \frac{1}{4^{m(1-\frac{k-j+1}{A-2})}} \left( \frac{\log x}{\log Q} \right)^{2j-1} & \text{if } 4^m > L_j, \\ 2L_j^{\frac{k-j+2}{A-2}} \int_{J_{2^m}} \frac{|L_{Q_0}^{(j)}(\sigma + it, f)|^2}{j!^2 (\log Q_0)^{2j-1}} \frac{dt}{t^2 + 1} & \text{if } 4^m \leq L_j. \end{cases}$$

so that

$$\begin{aligned} \sum_{m=1}^{\infty} \frac{I'_{j,2m}}{4^{m(1-\frac{k-j+1}{A-2})}} &\ll \frac{1}{L_j^{1-\frac{k-j+1}{A-2}}} \left( \frac{\log x}{\log Q} \right)^{2j-1} + L_j^{\frac{k-j+2}{A-2}} \int_{\mathbb{R}} \frac{|L_{Q_0}^{(j)}(\sigma + it, f)|^2}{j!^2 (\log Q_0)^{2j-1}} \frac{dt}{t^2 + 1} \\ &= \left( \frac{\log x}{\log Q} \right)^{\frac{(2j-1)(k-j+2)}{A-1}} \left( 1 + \int_{\mathbb{R}} \frac{|L_{Q_0}^{(j)}(\sigma + it, f)|^2}{j!^2 (\log Q_0)^{2j-1}} \frac{dt}{t^2 + 1} \right). \end{aligned}$$

Moreover,

$$(k-j+2)(2j-1) \leq \left( \left\lfloor \frac{k}{2} \right\rfloor + 1 \right) \left( 2k - 2 \left\lfloor \frac{k}{2} \right\rfloor + 1 \right) = \frac{(k+1)(k+2)}{2},$$

since  $k - \lfloor k/2 \rfloor + 1$  is the nearest integer to  $k/2 + 5/4$ . Hence relation (8.33) becomes

$$\begin{aligned} (8.34) \quad &\int_0^{\infty} \left| \frac{S_k(e^u; \Lambda f)}{k! e^{\sigma u}} \right|^2 du \\ &\ll \left( \frac{c_2 \log x}{\log y} \right)^{2k+1} + (c_{10} \log Q)^{2k+1} \left( \log \frac{\log x}{\log Q} \right)^{\alpha(A-2k-3)} + \frac{(c_{10} \log x)^{2k+1}}{T^2} \\ &\quad + (c_{10} \log Q_0)^{2k+1} \left( \frac{\log x}{\log Q} \right)^{\frac{(k+1)(k+2)}{2(A-1)}} \sum_{j=1}^{k+1} \left( 1 + \int_{\mathbb{R}} \frac{|L_{Q_0}^{(j)}(\sigma + it, f)|^2}{j!^2 (\log Q_0)^{2j-1}} \frac{dt}{t^2 + 1} \right). \end{aligned}$$

Lastly, for  $j \geq 1$  Plancherel's identity yields that<sup>5</sup>

$$(8.35) \quad \int_{\Re(s)=\sigma} |L_{Q_0}^{(j)}(\sigma + it, f)|^2 \frac{dt}{\sigma^2 + t^2} = 2\pi \int_{\log Q_0}^{\infty} \left| \frac{S_j(e^u; f_{Q_0})}{e^{\sigma u}} \right|^2 du,$$

where  $f_{Q_0}(n)$  is defined to be  $f(n)$  if  $P^-(n) > Q_0$  and 0 otherwise. By (8.19) and the inequality  $e^u \geq u^j/j!$ , we find that

$$S_0(e^u; f) \ll_{\delta} \frac{j! (\log Q_0)^j}{u^{j+2}} \quad (u \geq \log Q_0)$$

for some  $c_1 1 = c_1 1(\delta)$ , since  $1 \leq j \leq k+1 \leq (A-1)/2 \leq A-2$ . So Lemma 4.1(a) implies that

$$S_0(e^u; f_{Q_0}) \ll_{\delta} \frac{j! (c_{11} \log Q_0)^{j+1}}{u^{j+2}} \leq \frac{j! (c_{11} \log Q_0)^j}{u^{j+1}} \quad (u \geq \log Q_0).$$

Consequently,

$$S_j(e^u; f_{Q_0}) = O(e^{u/2} u^j) + \int_{u/2}^u w^j dS_0(e^w; f_{Q_0}) \ll \frac{(j+1)! (c_{11} \log Q_0)^j e^u}{u} \quad (u \geq \log Q_0),$$

by integration by parts. Inserting this estimate into (8.35), we deduce that

$$\int_{\Re(s)=\sigma} |L_{Q_0}^{(j)}(\sigma + it, f)|^2 \frac{dt}{\sigma^2 + t^2} \ll_{\delta} (j+1)!^2 (c_{11} \log Q_0)^{2j-1}.$$

Combining the above inequality with (8.34) completes the proof of (8.25) and hence of part (a).

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<sup>5</sup>Note that if  $j \geq 1$ , then  $S_j(e^u; f_{Q_0}) = 0$  for all  $u \leq \log Q_0$ .

(b) By arguing as in the proof of Lemma 4.2(b), we find that  $L(1, f) = 0$ . The claimed estimate then follows by the argument leading to (8.25) on using Lemmas 4.1(b) and 8.4(b) in place of Lemmas 4.1(a) and 8.4(a), respectively.  $\square$

**8.3. Completion of the proofs.** We conclude the paper with the promised proof of Theorems 1.2 and 2.1.

*Proof of Theorem 1.2.* Part (a) of Theorem 1.2 is an immediate consequence of Theorem 8.5(a) with  $T = (\log x)^{A-2}$  and  $k = \lfloor A - 1 \rfloor$ . For part (b), note that the function  $g(n) = f(n)n^{-it_0}$  satisfies (2.1) with  $\log Q \asymp (|t_0| + 1)^{\frac{1}{A-2}}$  and  $\delta = 1/2$ , by partial summation. So Theorem 8.5(b) with  $k = \lfloor A - 2 \rfloor$  completes the proof of the theorem.  $\square$

*Proof of Theorem 2.1.* (a) Let  $x \geq 100$ ,  $T \geq 1$  and  $k \in \mathbb{Z} \cap [0, (A-3)/2]$ . Set  $\sigma = 1 + 1/\log x$  and define  $y$  and  $Y$  by  $\log y = \min\{e^{N(x;T)}, \log x\}/(2\log Q + 2k)$  and  $\log Y = \log x/(2\log Q + 2k)$ . Note that it suffices to show the theorem when  $Q$  is large enough. Finally, we impose the condition  $2 \leq (\log y)^{k+\frac{1}{2}} \leq T$ , since the result is trivial when  $\log y < 2$  or  $T < 2$  and the case  $T > (\log y)^{k+\frac{1}{2}}$  follows by the case  $T = (\log y)^{k+\frac{1}{2}}$ .

First, we show the claimed estimate for  $\sum_{n \leq x} \mu(n)f(n)$ . Let  $\Delta = x/(\log y)^{k+1} \geq \sqrt{x}$ . Since  $|S_1(x; \mu f) - S_1(t; \mu f)| \leq \Delta \log x$  for  $t \in [x - \Delta, x]$ , we find that

$$|S_1(x; \mu f)| = \frac{1}{\Delta} \int_{x-\Delta}^x |S_1(t; \mu f)| dt + O(\Delta \log x).$$

Moreover, Lemma 8.1 with  $\mu f$  in place of  $f$ ,  $k = 1$  and  $r = k - 1$  implies that

$$|S_1(x; \mu f) - (\log x)S_0(x; \mu f)| \ll \frac{3^k x}{(\log x)^k} I_k(\sigma; \mu f) + 2^k \sqrt{x}.$$

Therefore

$$(8.36) \quad |S_0(x; \mu f)| = \frac{1}{\Delta \log x} \int_x^{x+\Delta} |S_1(t; \mu f)| dt + O\left(\frac{2^k x}{(\log y)^{k+1}} + \frac{3^k x}{(\log x)^{k+1}} I_k(\sigma; \mu f)\right).$$

Note that  $\mu f \log = -\mu f * \Lambda f$ , since  $\mu \log = -\mu * \Lambda$  and  $f$  is completely multiplicative. So Dirichlet's hyperbola method yields

$$(8.37) \quad \begin{aligned} S_1(t; \mu f) &= - \sum_{dm \leq t} \Lambda(d)f(d)\mu(m)f(m) \\ &= - \sum_{d \leq \sqrt{t}} \Lambda(d)f(d)S_0(t/d; \mu f) - \sum_{m \leq \sqrt{t}} \mu(m)f(m)S_0(t/m; \Lambda f) \\ &\quad + S_0(\sqrt{t}; \mu f)S_0(\sqrt{t}; \Lambda f). \end{aligned}$$

For  $u \in [\sqrt{x}, x]$  we have that

$$(8.38) \quad \begin{aligned} \frac{|S_0(u; \mu f)| + |S_0(u; \Lambda f)|}{u} &\ll c_1^k k! \left( \frac{1}{(\log y)^k} + \frac{\sqrt{(\log x)(\log \log Y)}}{(\log Y)^{k+\frac{1}{2}}} + \sqrt{\frac{\log x}{T}} \right) \\ &\ll c_1^k k! \sqrt{\log x} \left( \frac{\sqrt{\log \log y}}{(\log y)^{k+\frac{1}{2}}} + \frac{1}{T} \right) \end{aligned}$$

for some  $c_1 = c_1(\delta)$ , by Theorem 8.5 with  $2k + 1 \leq A - 2$  in place of  $k$  if  $k \geq 1$  and trivially if  $k = 0$ . So (8.37) becomes<sup>6</sup>

$$S_1(t; \mu f) = - \sum_{(\log y)^{2k+2} < d \leq \sqrt{t}} \Lambda(d) f(d) S_0(t/d; \mu f) - \sum_{(\log y)^{2k+2} < m \leq \sqrt{t}} \mu(m) f(m) S_0(t/m; \Lambda f) \\ + O \left( t c_2^k k! (\log \log y) \sqrt{\log x} \left( \frac{\sqrt{\log \log y}}{(\log y)^{k+\frac{1}{2}}} + \sqrt{\frac{1}{T}} \right) \right) \quad (x/2 \leq t \leq x)$$

for some  $c_2 = c_2(\delta)$ . Inserting this formula into (8.36), we deduce that

$$(8.39) \quad |S_0(x; \mu f)| \leq \frac{1}{\Delta \log x} \int_{x-\Delta}^x \sum_{(\log y)^{2k+2} < d \leq \sqrt{t}} (\Lambda(d) |S_0(t/d; \mu f)| + |S_0(t/d; \Lambda f)|) dt \\ + O \left( x c_3^k \left( \frac{k!}{(\log y)^{k+\frac{1}{2}}} + \frac{I_k(\sigma; \mu f)}{(\log x)^{k+1}} + \frac{k!}{\sqrt{T}} \right) \right)$$

for some  $c_3 = c_3(\delta)$ . Next, we have that

$$\int_{x-\Delta}^x \sum_{(\log y)^{2k+2} < d \leq \sqrt{t}} \Lambda(d) |S_0(t/d; \mu f)| dt = \sum_{(\log y)^{2k+2} < d \leq \sqrt{x}} \Lambda(d) d \int_{\frac{x-\Delta}{d}}^{\frac{x}{d}} |S_0(t; \mu f)| dt \\ \leq \int_{\frac{\sqrt{x}}{2}}^{\frac{x}{(\log y)^{2k+2}}} |S_0(t; \mu f)| \sum_{\frac{x-\Delta}{t} < d \leq \frac{x}{t}} d \Lambda(d) dt.$$

By the Brun-Titchmarsh inequality, we have that

$$\sum_{\frac{x-\Delta}{t} < d \leq \frac{x}{t}} d \Lambda(d) \ll \frac{x}{t} \left( \sqrt{\frac{x}{t}} + \log \frac{x}{t} \sum_{\frac{x-\Delta}{t} < p \leq \frac{x}{t}} 1 \right) \ll \frac{\Delta x}{t^2},$$

uniformly for  $t \leq x/(\log y)^{2k+2}$ , since  $\Delta/t \geq \sqrt{x/t}$  in this range. So we deduce that

$$(8.40) \quad \int_{x-\Delta}^x \sum_{(\log y)^{2k+2} < d \leq \sqrt{t}} \Lambda(d) |S_0(t/d; \mu f)| \ll \Delta x \int_{\frac{\sqrt{x}}{2}}^x \frac{|S_0(t; \mu f)|}{t^2} dt.$$

Moreover, partial summation implies that

$$S_0(t; \mu f) = O(\sqrt{t}) + \int_{\sqrt{t}}^t \frac{1}{(\log u)^k} dS_k(u; \mu f) = O(\sqrt{t}) + \frac{S_k(t; \mu f)}{(\log t)^k} + \int_{\sqrt{t}}^t \frac{k S_k(u; \mu f)}{u (\log u)^{k+1}} du,$$

which, together with (8.40) and the Cauchy-Schwarz inequality, implies that

$$\int_{x-\Delta}^x \sum_{(\log y)^{2k} < d \leq \sqrt{t}} \Lambda(d) |S_0(t/d; \mu f)| \ll \Delta x \left( \frac{5^k}{(\log x)^k} \int_{\frac{\sqrt{x}}{2}}^x \frac{|S_k(t; \mu f)|}{t^2} dt + \frac{1}{x^{1/4}} \right) \\ \ll \Delta x \left( \frac{5^k I_k(\sigma; \mu f)}{(\log x)^{k-1/2}} + \frac{1}{x^{1/4}} \right).$$

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<sup>6</sup>We estimate the term  $S_0(\sqrt{t}; \mu f) S_0(\sqrt{t}; \Lambda f)$  by applying (8.38) to  $S_0(\sqrt{t}; \Lambda f)$  and by bounding  $|S_0(\sqrt{t}; \mu f)|$  trivially by  $\sqrt{t}$ .

Similarly, we find that

$$\int_{x-\Delta}^x \sum_{(\log y)^{2k+2} < d \leq \sqrt{t}} |S_0(t/d; \Lambda f)| \ll \Delta x \left( \frac{5^k I_k(\sigma; \Lambda f)}{(\log x)^{k-\frac{1}{2}}} + \frac{1}{x^{1/4}} \right).$$

Combining the two above estimates with (8.39) implies that

$$\frac{|S_0(x; \mu f)|}{x} \ll c_4^k \left( \frac{k!}{(\log y)^{k+\frac{1}{2}}} + \frac{I_k(\sigma; \mu f)}{(\log x)^{k+\frac{1}{2}}} + \frac{I_k(\sigma; \Lambda f)}{(\log x)^{k+\frac{1}{2}}} + \frac{k!}{\sqrt{T}} \right)$$

for some  $c_4 = c_4(\delta) > 0$ . Estimating  $I_k(\sigma; \mu f)$  and  $I_k(\sigma; \Lambda f)$  by Theorem 8.6(a) completes the proof of (2.3).

Finally, we show (2.5). Define  $z$  by

$$(8.41) \quad \log z = \max \left\{ \log Q_0, \frac{\log x}{\log y} \right\} \asymp_\delta \frac{\log x}{\log y},$$

where  $Q_0$  is given by (2.2). We may assume that  $y$  is large enough, so that  $z \leq x^{1/3}$ ; else, (2.5) follows trivially. Set  $g(n) = f(n)$  if  $P^-(n) > z$  and  $g(n) = 0$  otherwise. Relation (8.19) with  $t = 0$  and the inequality  $e^u \geq u^k/k!$  imply that

$$S_0(w; g) \ll_\delta w \cdot \frac{k!(\log z)^k}{(\log w)^{k+2}} \quad (w \geq z),$$

since  $k \leq A - 2$ . So Lemma 4.1(a) imply that

$$(8.42) \quad S_0(w; g) \ll_\delta w \cdot \frac{k!40^k(\log z)^{k+1}}{(\log w)^{k+2}} \leq w \cdot \frac{k!(40 \log z)^k}{(\log w)^{k+1}} \quad (w \geq z).$$

Moreover, similarly to (8.37), we have that

$$S_1(x; g) = \sum_{d \leq x^{1/4}} g(d) \Lambda(d) (S_0(x/d; g) - S_0(x^{3/4}; g)) + \sum_{m \leq x^{3/4}} g(m) S_0(x/m; \Lambda g).$$

The above formula, (8.41) and (8.42) yield that

$$\sum_{m \leq x^{1/4}} g(m) S_0(x/m; \Lambda g) \ll \frac{xc_6^k k!}{(\log y)^k}$$

for some  $c_6 = c_6(\delta) > 0$ . Since  $S_0(w; \Lambda g) = S_0(w; \Lambda f) + O(z \log w) = S_0(w; \Lambda f) + O(x^{1/3} \log w)$  for  $w \geq 1$ , we find that

$$\sum_{m \leq x^{1/4}} g(m) S_0(x/m; \Lambda f) \ll \frac{xc_7^k k!}{(\log y)^k}$$

for some  $c_7 = c_7(\delta) > 0$ . When  $1 < m \leq Z := \max\{z, (\log y)^{2k+2}\}$ , we use (8.38) to bound  $S_0(x/m; \Lambda f)$ . Since  $g(m) = 0$  for  $1 < m \leq z$  and  $\sum_{z < m \leq Z} |g(m)|/m \ll k(\log \log y)/\log z$ , we deduce that

$$\begin{aligned} S_0(x; \Lambda f) + \sum_{Z < m \leq x^{1/4}} g(m) S_0(x/m; \Lambda f) &\ll xc_8^k k! \frac{(\log \log y) \sqrt{\log x}}{\log z} \left( \frac{\sqrt{\log \log y}}{(\log y)^{k+\frac{1}{2}}} + \frac{1}{\sqrt{T}} \right) \\ &\quad + \frac{xc_7^k k!}{(\log y)^k} \end{aligned}$$



for some  $c_8 = c_8(\delta) > 0$ . Next, recall that  $\Delta = x/(\log y)^{k+1} \geq \sqrt{x}$  and note that  $|S_0(x/m; \Lambda f) - S_0(t/m; \Lambda f)| \ll \Delta/m$  for  $t \in [x-\Delta, x]$  and  $m \leq x^{1/4}$ . Since  $\sum_{m \leq x^{1/4}} |g(m)|/m \ll (\log x)/\log z \asymp_\delta \log y$ , by (8.41), we find that

$$\begin{aligned} S_0(x; \Lambda f) &= \sum_{Z < m \leq x^{1/4}} g(m) \cdot \frac{-1}{\Delta} \int_{x-\Delta}^x S_0(t/m; \Lambda f) dt + O \left( x c_9^k k! \left( \frac{(\log \log y)^{\frac{3}{2}}}{(\log y)^{k+\frac{1}{2}}} + \sqrt{\frac{\log x}{T}} \right) \right) \\ &= \frac{-1}{\Delta} \int_{\frac{x-\Delta}{x^{1/4}}}^{\frac{x}{Z}} S_0(t; \Lambda f) \sum_{\frac{x-\Delta}{t} < m \leq \frac{x}{t}} m g(m) dt + O \left( x c_9^k k! \left( \frac{(\log \log y)^{\frac{3}{2}}}{(\log y)^{k+\frac{1}{2}}} + \sqrt{\frac{\log x}{T}} \right) \right) \end{aligned}$$

for some  $c_9 = c_9(\delta)$ . For every  $t \in [\sqrt{x}, x/Z] \subset [(x-\Delta)/x^{1/4}, x/Z]$  we apply Lemma 3.3 with  $D = (x/t)^{1/3} \geq Z^{1/3}$  and  $z^{1/9}$  in place of  $y$  to obtain the estimate

$$\begin{aligned} \left| \sum_{\frac{x-\Delta}{t} < m \leq \frac{x}{t}} m g(m) \right| &\leq \frac{x}{t} \sum_{\substack{\frac{x-\Delta}{t} < m \leq \frac{x}{t} \\ P^-(m) > z^{1/9}}} 1 \leq \frac{x}{t} \sum_{\frac{x-\Delta}{t} < m \leq \frac{x}{t}} (\lambda^+ * 1)(m) \\ &= \frac{x}{t} \sum_{d \leq (x/t)^{1/3}} \lambda^+(d) \left( \frac{\Delta/t}{d} + O(1) \right) \ll \frac{\Delta x \log y}{t^2 \log x}, \end{aligned}$$

since  $\log z \asymp \log x / \log y$  and  $\Delta/t \geq \sqrt{x/t}$  for  $t \leq x/Z$ . Consequently,

$$(8.43) \quad \frac{S_0(x; \Lambda f)}{x} \ll \frac{\log y}{\log x} \int_{\sqrt{x}}^x \frac{|S_0(t; \Lambda f)|}{t^2} dt + c_8^k k! \left( \frac{(\log \log y)^{\frac{3}{2}}}{(\log y)^{k+\frac{1}{2}}} + \sqrt{\frac{\log x}{T}} \right).$$

Finally, the argument following (8.40) and Theorem 8.6(a) imply that

$$\int_{\sqrt{x}}^x \frac{|S_0(t; \Lambda f)|}{t^2} dt \ll \frac{5^k I_k(\sigma; \Lambda f)}{(\log x)^{k-\frac{1}{2}}} + \frac{1}{x^{1/4}} \ll \frac{c_{10}^k k! \log x}{(\log y)^{k+\frac{1}{2}-\frac{(k+1)(k+2)}{4(A-1)}}}$$

for some  $c_{10} = c_{10}(\delta) > 0$ . Inserting this estimate into (8.43) completes the proof of (2.5) and hence of part (a) of the theorem.

(b) By arguing as in the proof of Lemma 4.2(b), we find that  $L(1, f) = 0$ . The claimed result then follows by the argument leading to (2.3) (with  $T = \infty$ ,  $e^k Q$  in place of  $z$  and  $g(n) = (1 * f)(n)$  whenever  $P^-(n) > z$  and  $g(n) = 0$  otherwise), on using Lemma 4.1(b), Theorem 8.5(b) and Theorem 8.6(b) in place of Lemma 4.1(a), Theorem 8.5(a) and Theorem 8.6(a), respectively. Note that since  $1 \leq k \leq (A-3)/2$ , we must have that  $A \geq 5$  and, consequently,  $\alpha(A-k-3) \geq \alpha((A-3)/2) = 0$ .  $\square$

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